Linear and nonlinear representations of wave fields and their application to processing of radio occultations

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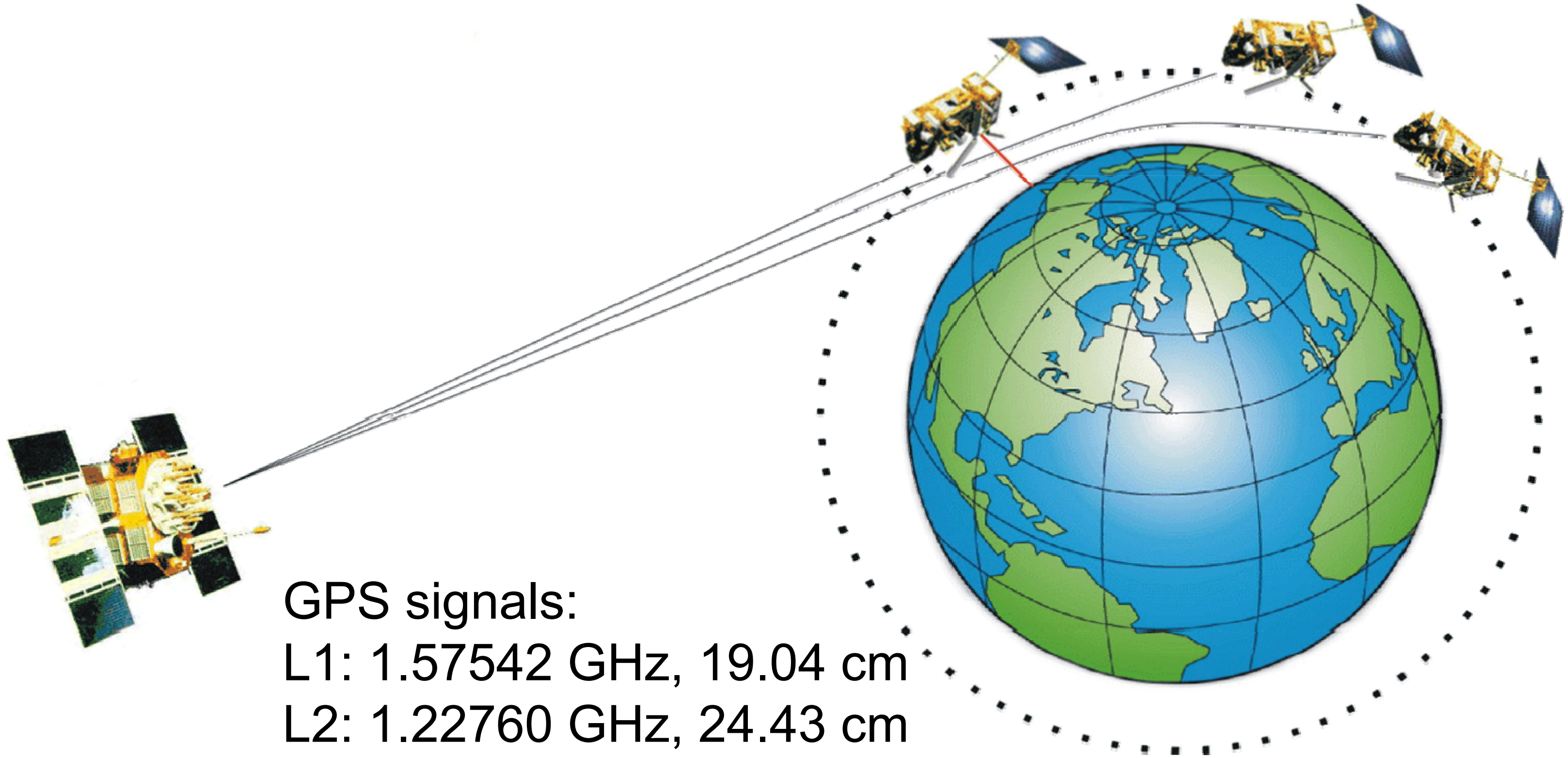
Contents

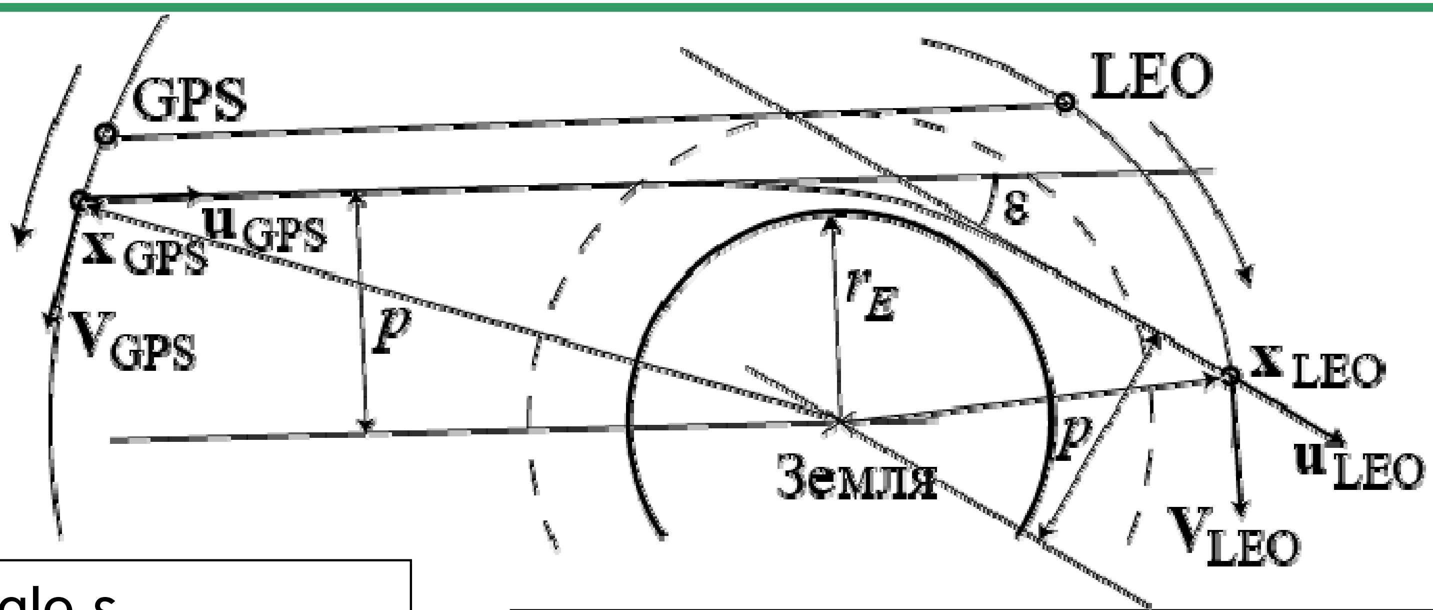
- Radio occultation sounding of the Earth's atmosphere
- The problem of the interpretation of measured wave fields
- Linear representations of wave fields and the reconstruction of the ray manifold
- Approach to the reconstruction of generic structure of ray manifold: definition of 2D density in the phase space
- Winger distribution function (WDF): definition and properties
- Application for radio occultations

Radio occultations



1. Measurements of refraction of radio signals of GNSS (GPS, GLONASS, Galileo) in the Earth's atmosphere on limb paths.
2. Retrieval of vertical profiles of temperature, pressure and humidity.

Radio occultations



Bending angle ε
Impact parameter *p*

$$\varepsilon(p) \xleftrightarrow{\text{Linear transformation}} \ln n(x)$$

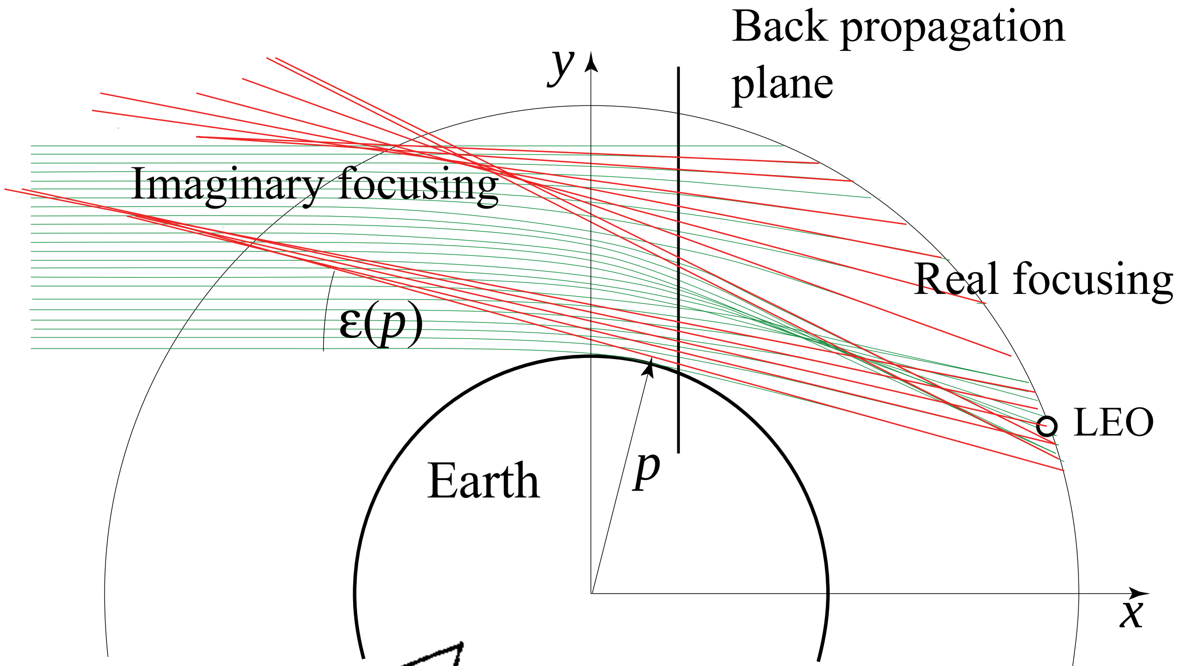
Refractivity *n*
Distance from the Earth's center *r*
Refractive radius *x*

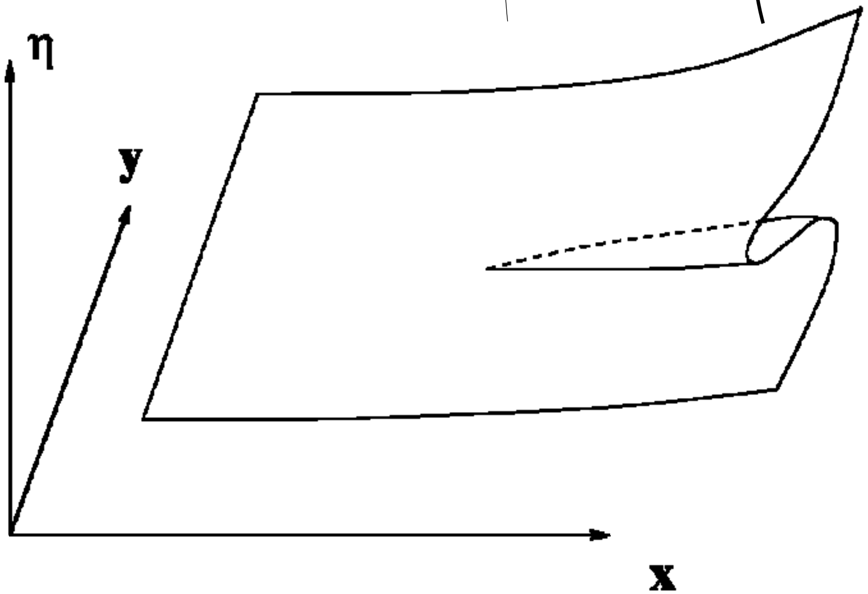
$$\varepsilon(p) = -2p\int_{r_0}^{\infty} \frac{d\ln n(r)}{dr}\frac{dr}{\sqrt{n(r)^2 r^2 - p^2}}$$

$$n(x) = \exp\left(\frac{1}{\pi}\int_{p}^{\infty}\frac{\varepsilon(p)dp}{\sqrt{p^2 - x^2}}\right)$$

$$x = n(r)r; \quad r(x) = \frac{x}{n(x)}$$

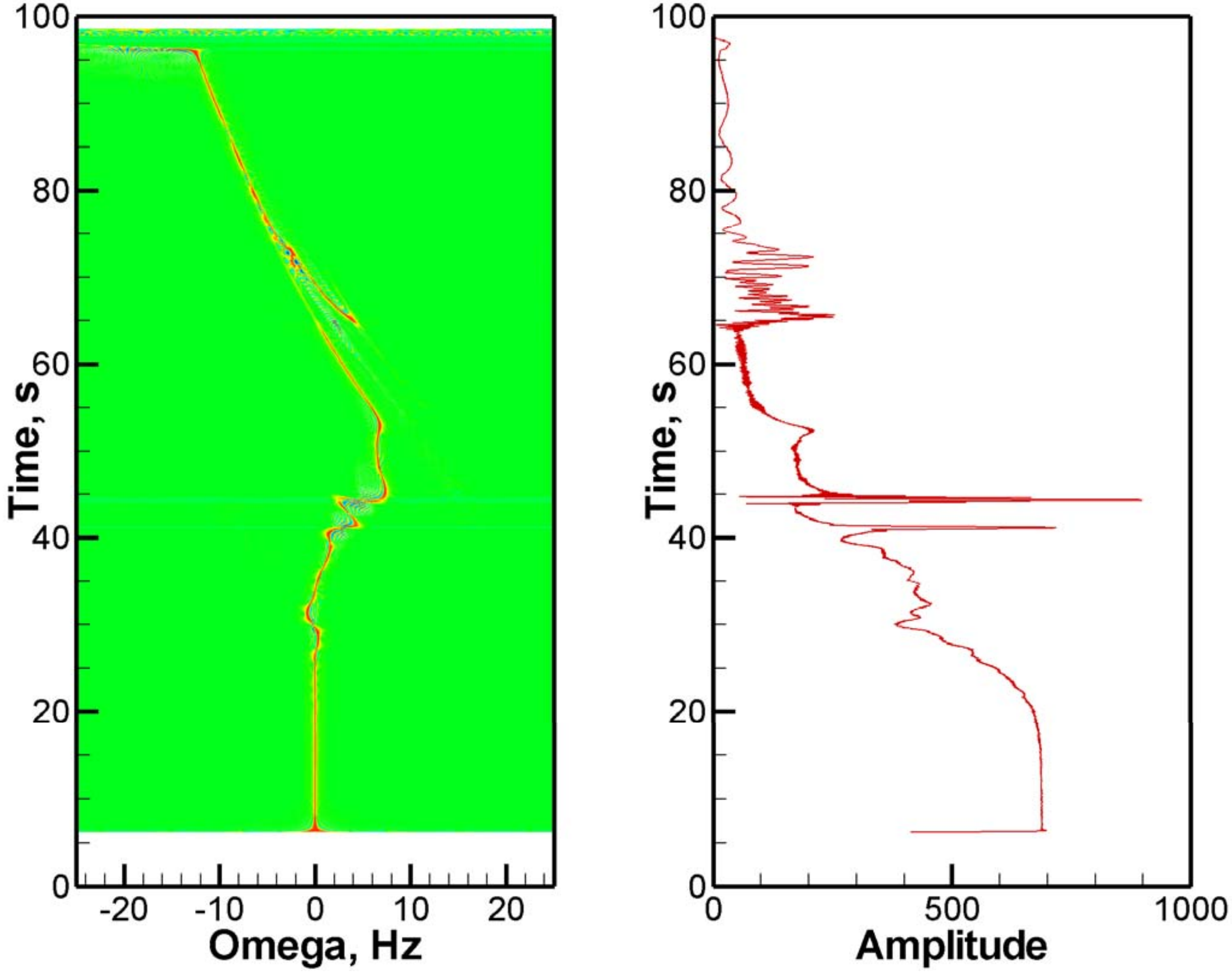
Multipath propagation





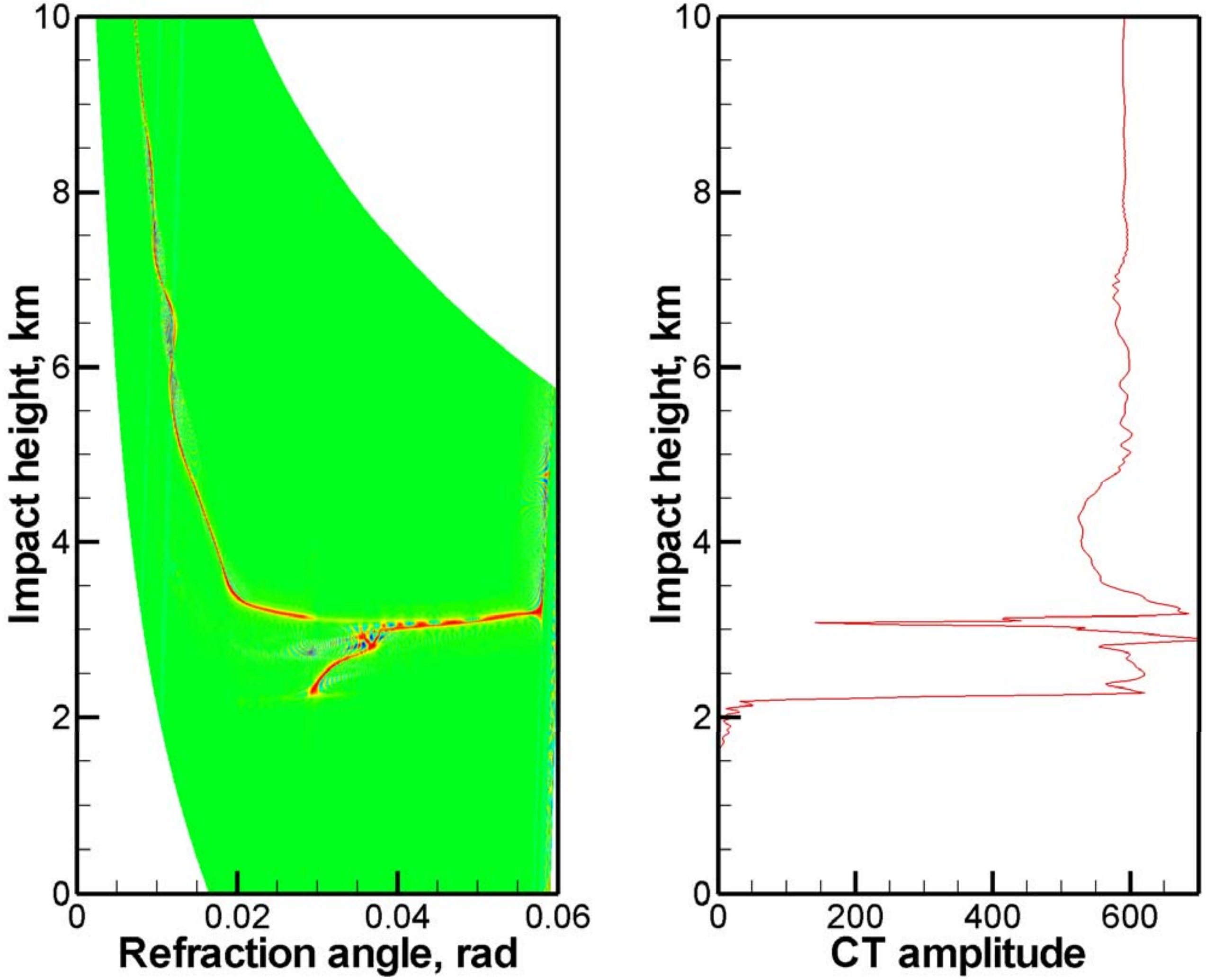
For the retrieval of ray manifold structure in multipath zones, we used approaches introduced by V.A.Fock, V.P.Masolv, Yu.V.Egorov, and L. Hörmander and developed *Canonical Transform method*.

Ray manifold and its projections



$$A(t) = \frac{\Delta E}{\Delta t}$$ Amplitude is the energy density wrt time

Ray manifold and its projections



$$A'(p) = \frac{\Delta E}{\Delta p}$$ CT amplitude is the energy density wrt impact height

Canonical Transforms

Wave field in t- and p-representations:

$$u(t) = \sum A_i(p)\exp\left[ik\int \sigma_i(t)\,dt\right],$$

$$\hat{u}(p) = \sum A_i'(p)\exp\left[ik\int \varepsilon_i(p)\,dp\right]$$

$k = \frac{2\pi}{\lambda}$ - wave number

$\sigma = -\frac{\omega}{k}$ - optical path derivative

Linear transformation between the representations

$$\hat{u}(p) = \int K(p,t)\,u(t)\,dt$$

corresponds to canonical transform

$$(t,\sigma) \to (p,\varepsilon)$$

which conserves volume element:

$$dt \wedge d\sigma = dp \wedge d\varepsilon$$

$$\varepsilon\,dp - \sigma\,dt = dS(p,t) \qquad K(p,t) = \sqrt{\mu(p,t)\frac{\partial^2 S(p,t)}{\partial p\,\partial t}}\exp\left(ikS(p,t)\right)$$

References

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2. V.P. Maslov, Perturbation theory and asymptotical methods, Moscow: Moscow State University, 1965.
3. Yu.V. Egorov, About canonical transforms of pseudo-differential operators, Advances in Mathematics, 1969, V. 24, No. 5, 149—150.
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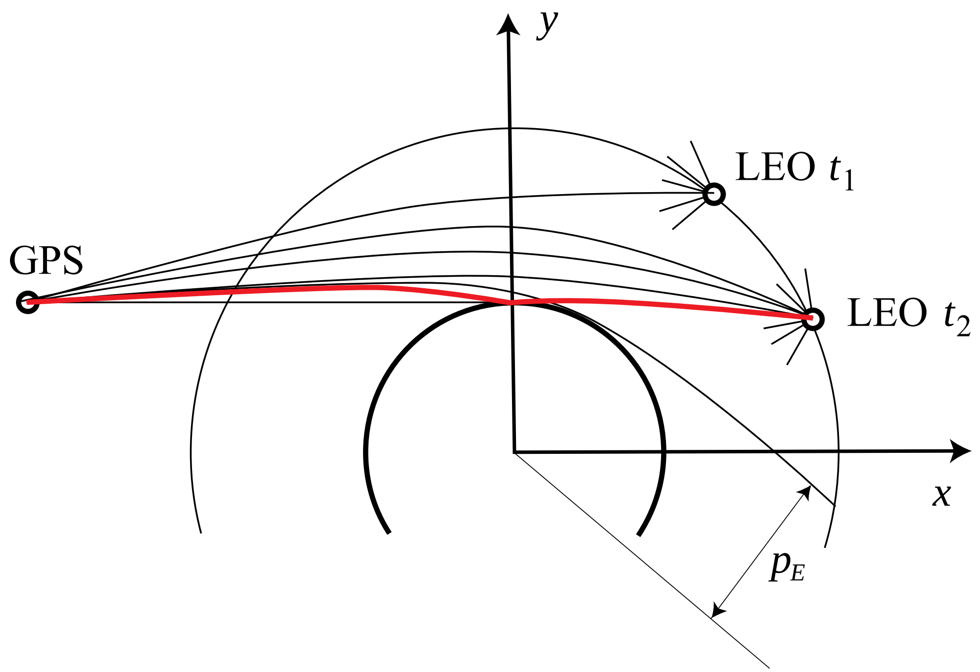
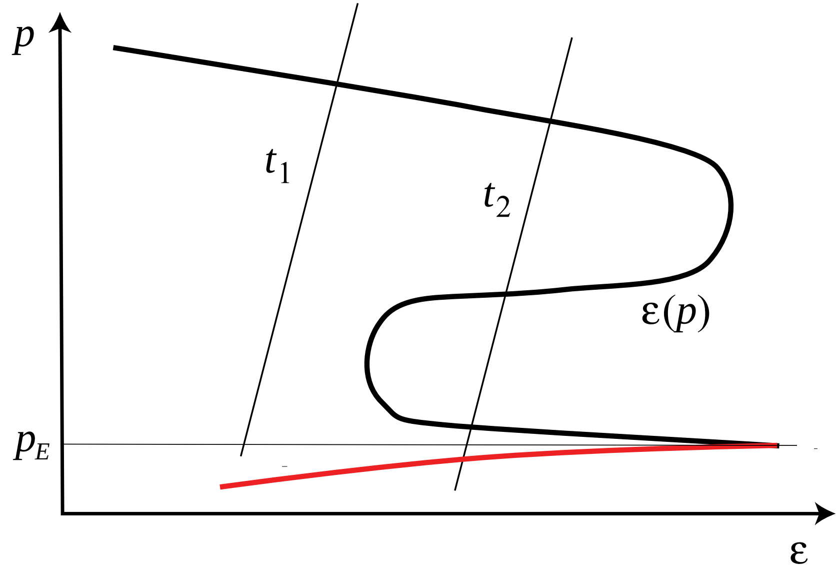
Density in phase space

- Linear representations of wave field are linked to different projections of the ray manifold.
- For the correct retrieval of the ray manifold structure it is necessary to find a single-valued projection. Such a projection may be unknown in advance. For radio occultations, projection to the impact parameter axis is single-valued for a spherically-symmetric atmosphere, but this property may be violated in presence of horizontal gradients.
- We will now define an universal energy density in 2D phase space, which is not linked to any specific coordinate choice.
- An example of such density is spectrogram (radio holographic sliding spectra). Its disadvantage is the limited resolution due to the uncertainty relation.

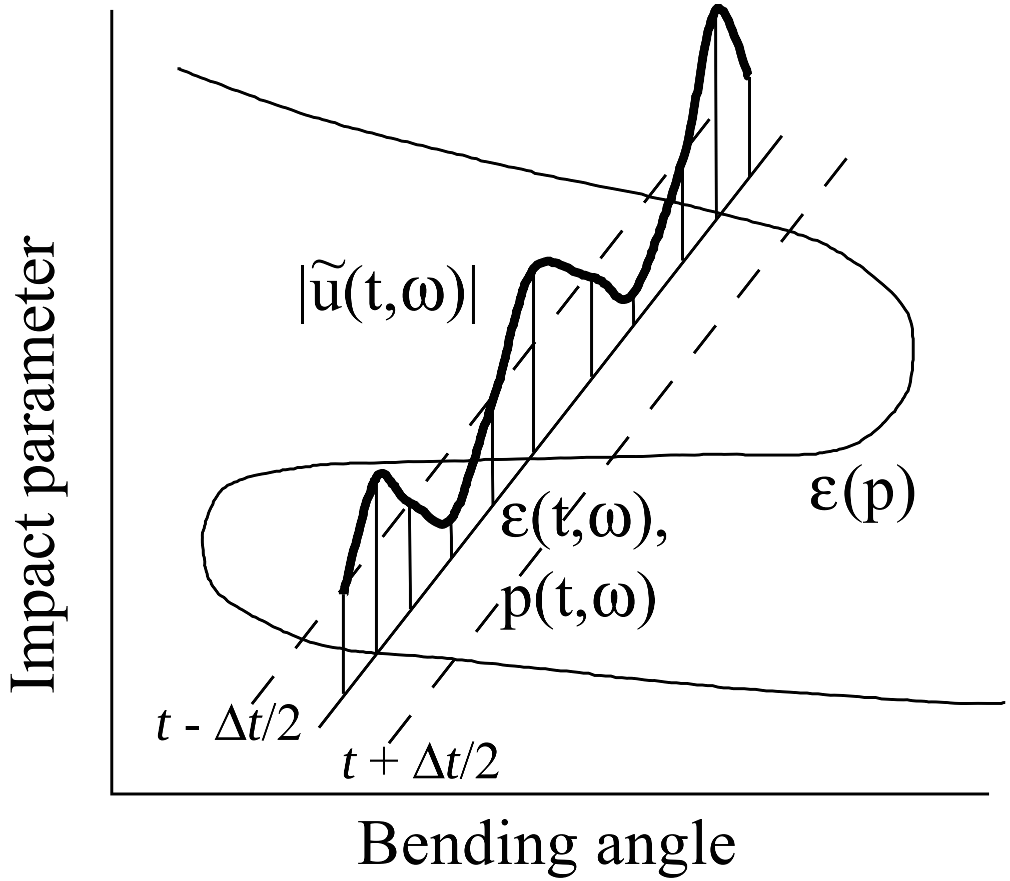
Radio holographic analysis





The manifold of all virtual rays that can be received in the given point form a line in the ray coordinate plane (ε,p). The intersections of this line with bending angle profile correspond to actual rays being received. **Red color** marks the reflected ray and the fragment of the bending angle profile corresponding to reflected rays.

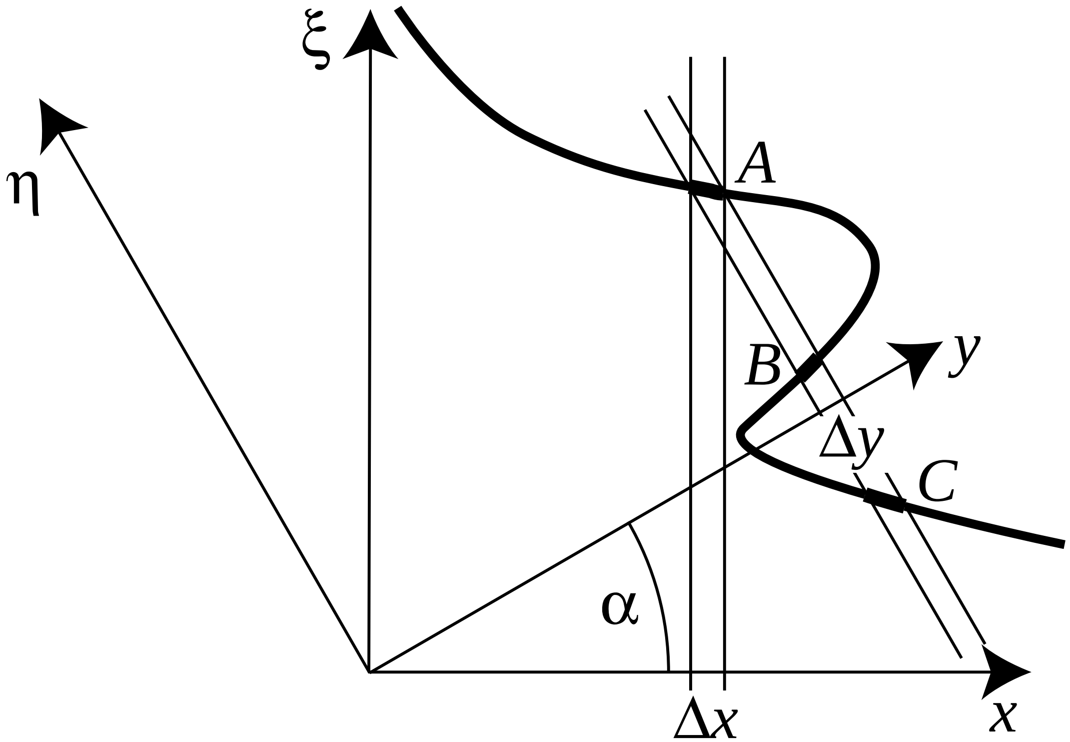
Radio holographic analysis



At every moment of time we compute a sliding spectrum of the signal in the aperture from t-Δt/2 to t+Δt/2. Each Doppler frequency correspond to some ray direction. Maxima of the spectrum correspond to observed rays.

Uncertainty relation: $\Delta t\ \Delta\omega \sim 2\pi$ or $\Delta p\ \Delta\varepsilon \sim \lambda$

Tomography



$$|u_\alpha(y)|^2 \Delta y = \sum_{A,B,C,\ldots} \Delta E_i$$ Tomographic equation

$$\Delta E = W(x,\xi)\,\Delta x\,\Delta\xi$$ Energy distribution W

$$u_\alpha(y) = \int K_\alpha(y,x)\,u(x)\,dx$$ $u_\alpha(y)$ is a linear representation of u(x)

Tomography

$$W(x,\xi) = \frac{1}{2\pi} \int u\left(x - \frac{s}{2}\right) \bar{u}\left(x + \frac{s}{2}\right) \exp(i\xi s)\, ds$$

The solution of the tomographic equation is
Wigner distribution function (WDF).

M. E. Gorbunov and K. B. Lauritsen, Analysis of wave fields by Fourier Integral Operators and its application for radio occultations, Radio Science, 2004, 39(4), RS4010, doi:10. 1029/2003RS002971.

M. E. Gorbunov, K. B. Lauritsen, and S. S. Leroy, Application of Wigner distribution function for analysis of radio occultations, Radio Science , 2010, 2010RS004388, in press.

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1. P.A.M. Dirac, Note on exchange phenomena in the Thomas atom, Proc. Camb. Phil. Soc. 26, 376-395 (1930).
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3. W. Heisenberg, Über die inkohärente Streuung von Röntgenstrahlen, Physik. Zeitschr. 32, 737-740 (1931).
4. E.P. Wigner, On the quantum correction for thermodynamic equilibrium, Phys. Rev. 40, June 1932, 749-759.
5. J. Ville, Théorie et Applications de la Notion de Signal Analytique, Cables et Transmission, 2A: (1948) 61-74.

Properties of WDF

1. WDF is real.
2. WDF is not necessary positive, but its integral over the phase plane equals the full energy.
3. WDF consists of two components: positive component in the vicinity of ray manifold (tends to the micro-canonical distribution) and oscillation interference component (quantum oscillations, cross-terms).
4. WDF is a non-linear representation of the wave fields, i.e. it contains full information about the wave field. Wave field can be retrieved from WDF up to a constant phase shift.
5. Accurate determination of the linear trend of frequency:

$$u(x) = \exp\left(iax + ibx^2 / 2\right)$$

$$\xi = a + bx$$

$$W(x,\xi) = \delta\left(\xi - (a + bx)\right)$$

Properties of WDF

6. Composition of two fields:

$$u(x) = \exp(ia_1 x) + \exp(ia_2 x)$$

$$W(x,\xi) = \delta(\xi - a_1) + \delta(\xi - a_2) + 2\cos((a_1 - a_2)x)$$

$$\Delta x \Delta a \sim 2\pi$$

The sum of two waves, the two ray manifold (two δ-functions) in the phase space are accurately reproduced, but interference terms emerge. The period of the interference oscillations is a function of the momentum difference as follows from the uncertainty relation.

7. WDF can be used for processing arbitrary oscillating signals consisting of multiple sub-signals with gliding frequencies. It is possible to consider a cross-density of two signals:

$$W_{12}(x,\xi) = \frac{1}{2\pi} \int u_1\left(x - \frac{s}{2}\right) \overline{u}_2\left(x + \frac{s}{2}\right) \exp(i\xi s)\, ds$$

Processing of radio occultations

$\mathbf{r}_G(t), \quad \mathbf{r}_L(t)$	coordinates of transmitter (GPS) and receiver (LEO)
$A(t)$	amplitude
$S_0(t) = \lvert \mathbf{r}_G(t) - \mathbf{r}_L(t) \rvert$	vacuum phase delay
$S(t)$	atmospheric phase excess
$\omega = -k\left(\dot{S}_0 + \dot{S}\right)$	Doppler frequency
$S_M(t)$	phase model (predicting Doppler frequency within 10–15 Hz)

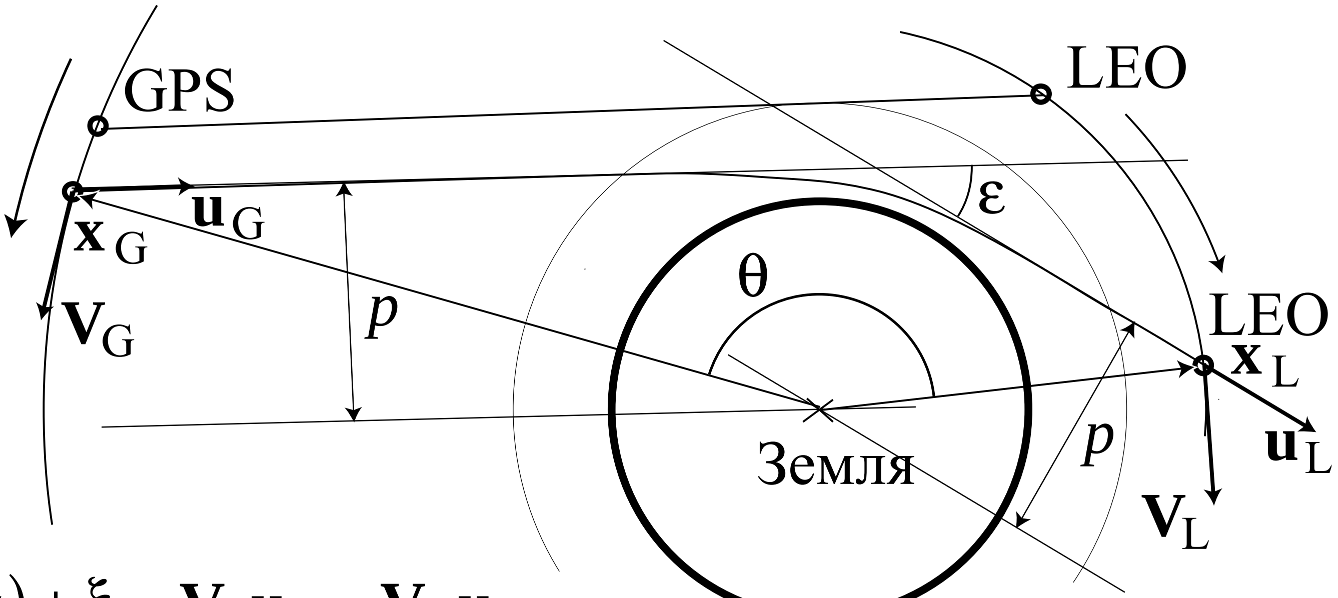
$u(t) = A(t)\exp\left(ik\left(S(t) - S_M(t)\right)\right)$ wave field with down-converted frequency

WDF and Weighted WDF

$$W(t,\xi) = \frac{k}{2\pi}\int \exp(ik\xi s)\, u\left(t - \frac{s}{2}\right)\bar{u}\left(t + \frac{s}{2}\right) ds$$

$$\tilde{W}(t,\xi) = \frac{k}{2\pi}\int \exp(ik\xi s)\, u\left(t - \frac{s}{2}\right)\bar{u}\left(t + \frac{s}{2}\right)\exp\left(-k^2\Delta\xi^2 s^2\right) ds$$

Processing of radio occultations



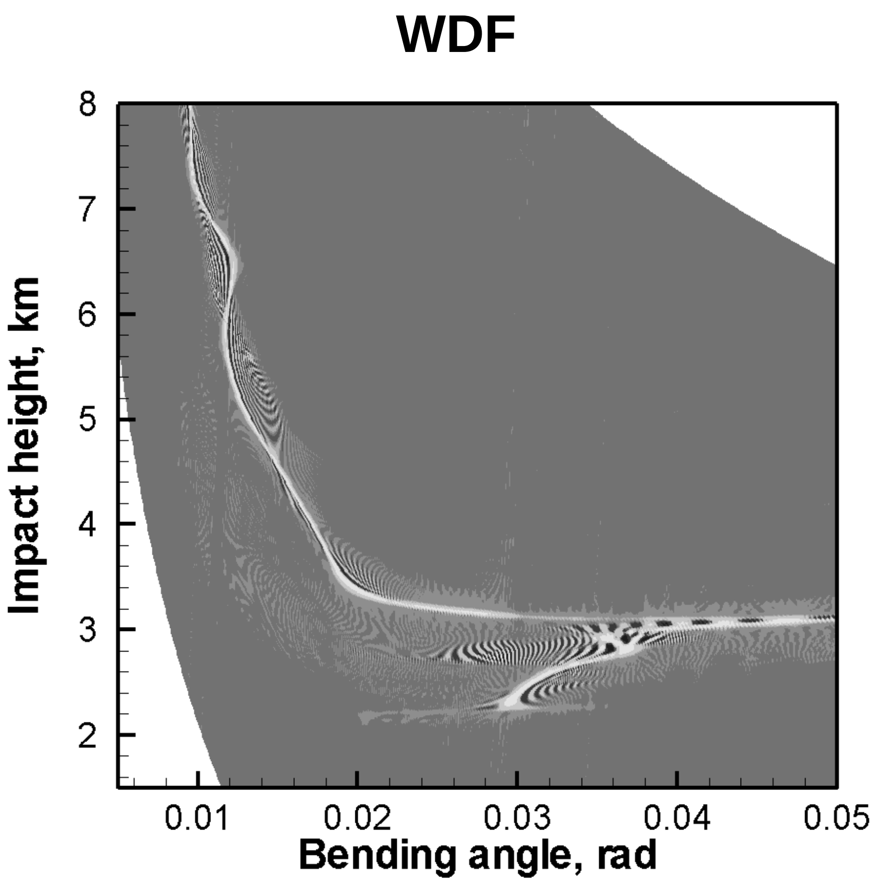
$$\dot{S}_0(t) + \dot{S}_M(t) + \xi = \mathbf{V}_L \mathbf{u}_L - \mathbf{V}_G \mathbf{u}_G =$$

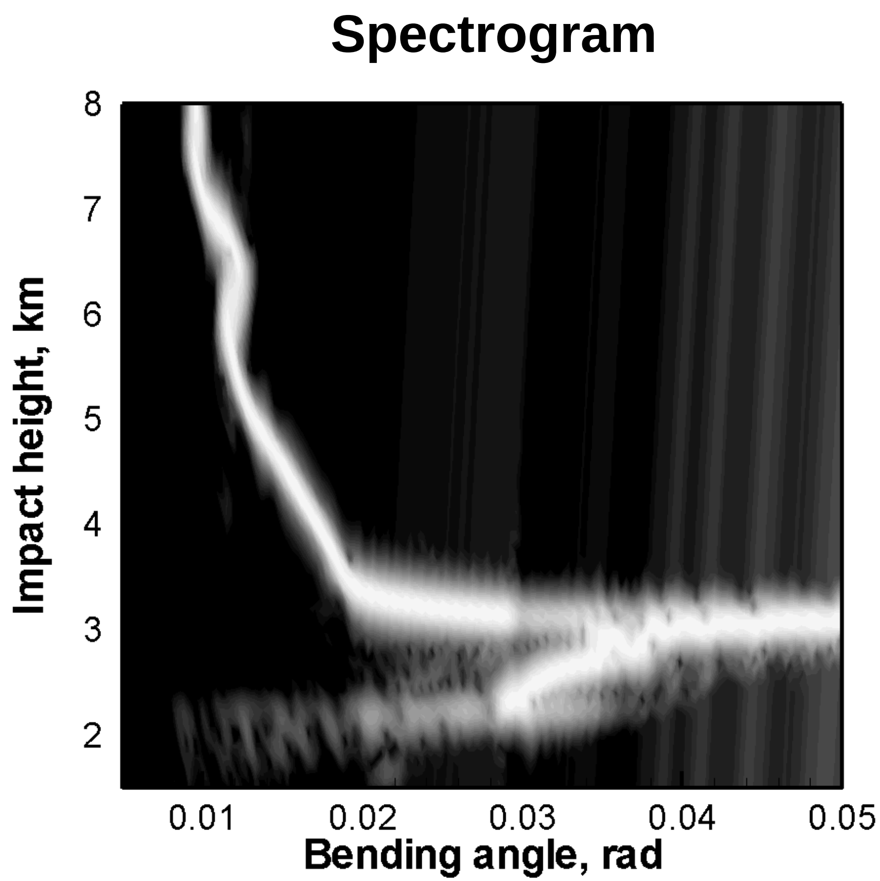
$$= \dot{\theta}(t)\, p(t,\xi) + \frac{\dot{r}_L(t)}{r_L(t)} \sqrt{r_L^2(t) - p(t,\xi)^2} + \frac{\dot{r}_G(t)}{r_G(t)} \sqrt{r_G^2(t) - p(t,\xi)^2}$$

$$\varepsilon(t,\xi) = \theta(t) - \arccos\frac{p(t,\xi)}{r_L(t)} - \arccos\frac{p(t,\xi)}{r_G(t)}$$

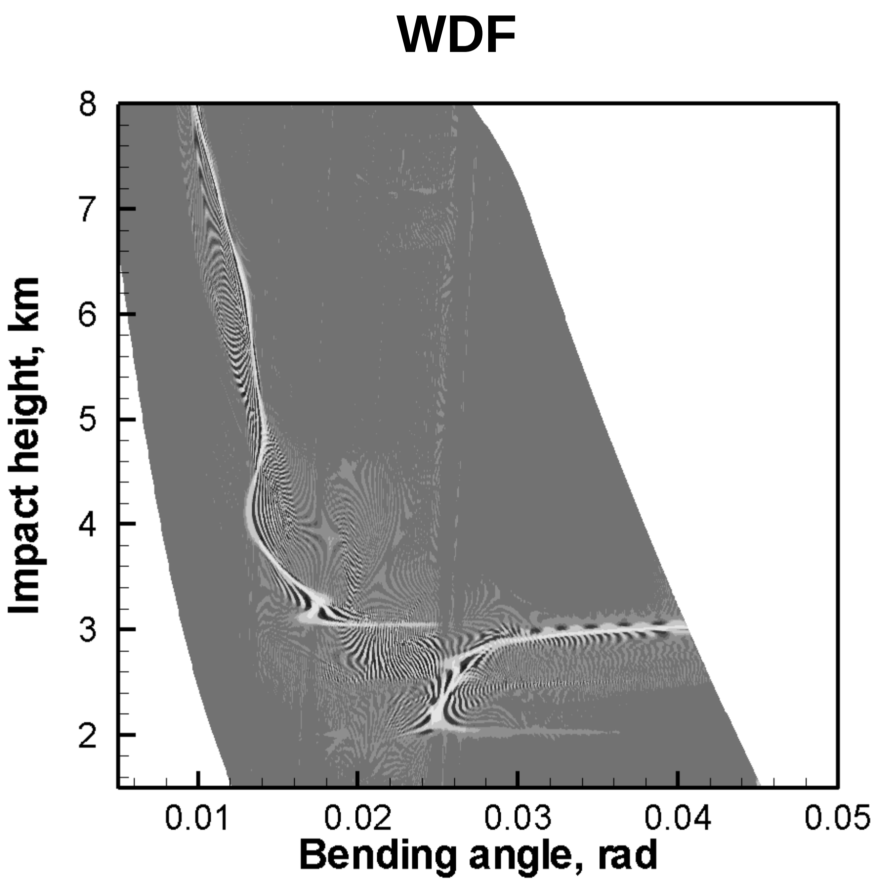
$W(t,\xi) \rightarrow W(p,\varepsilon)$	WDF can be looked at as a function of impact parameter and bending angle

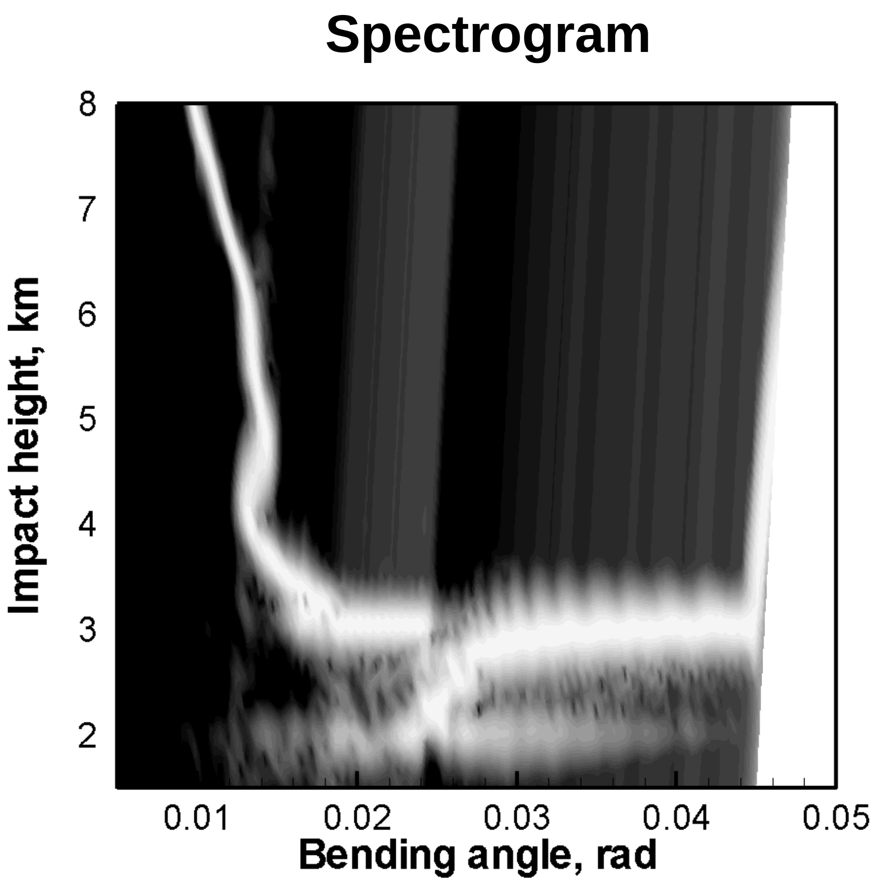
Processing artificial data





Processing artificial data





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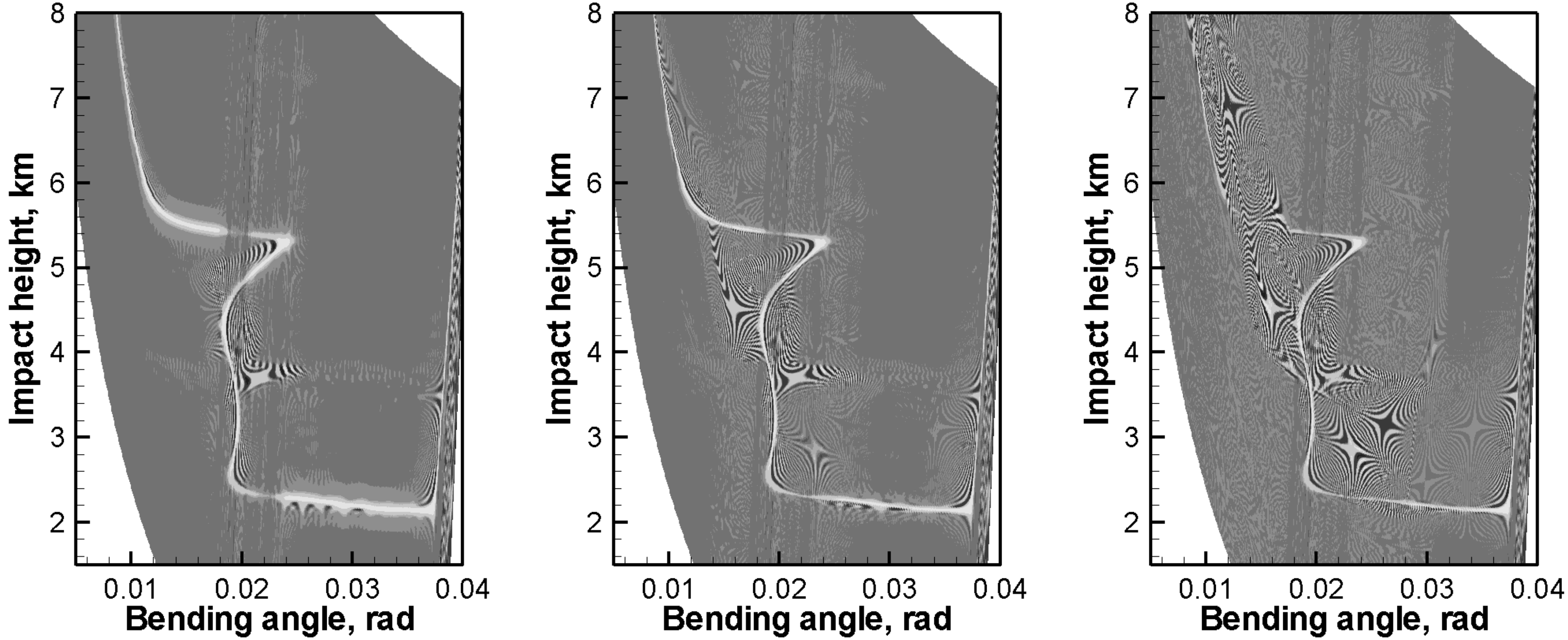
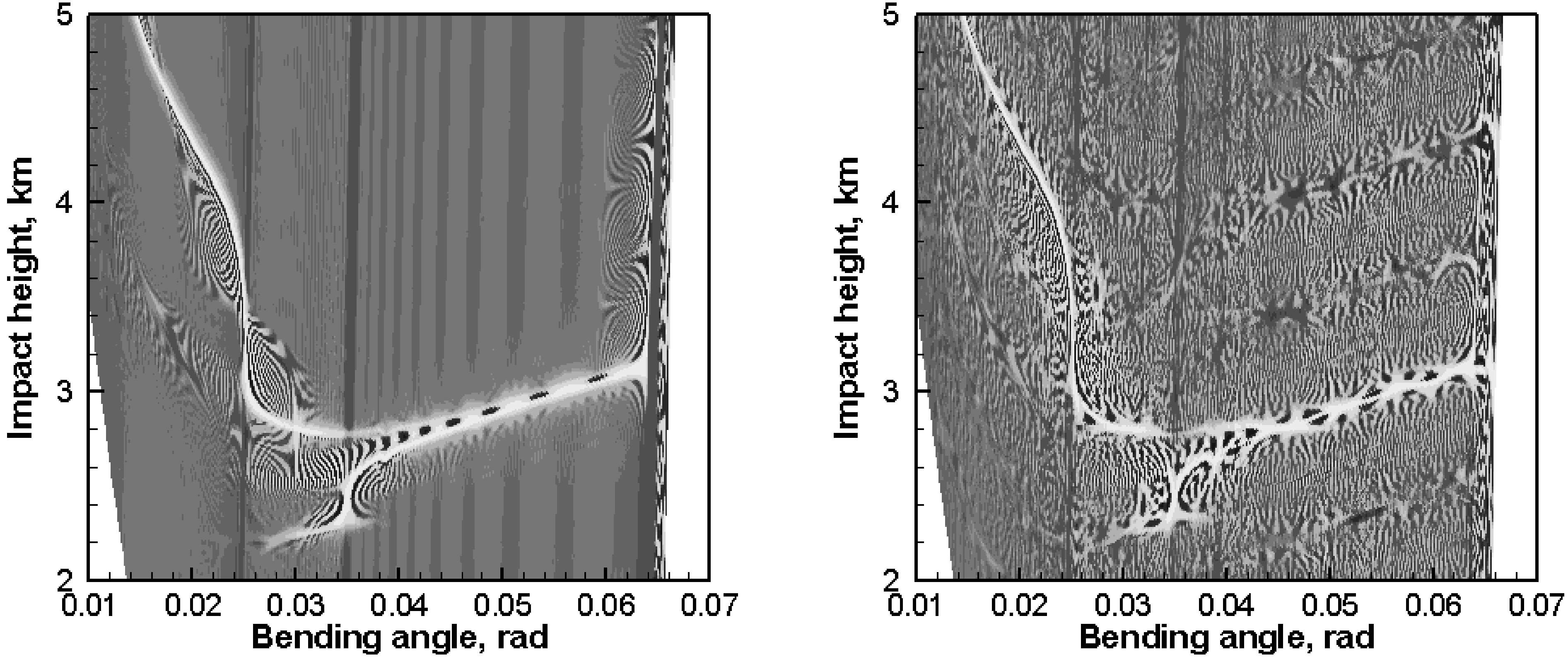


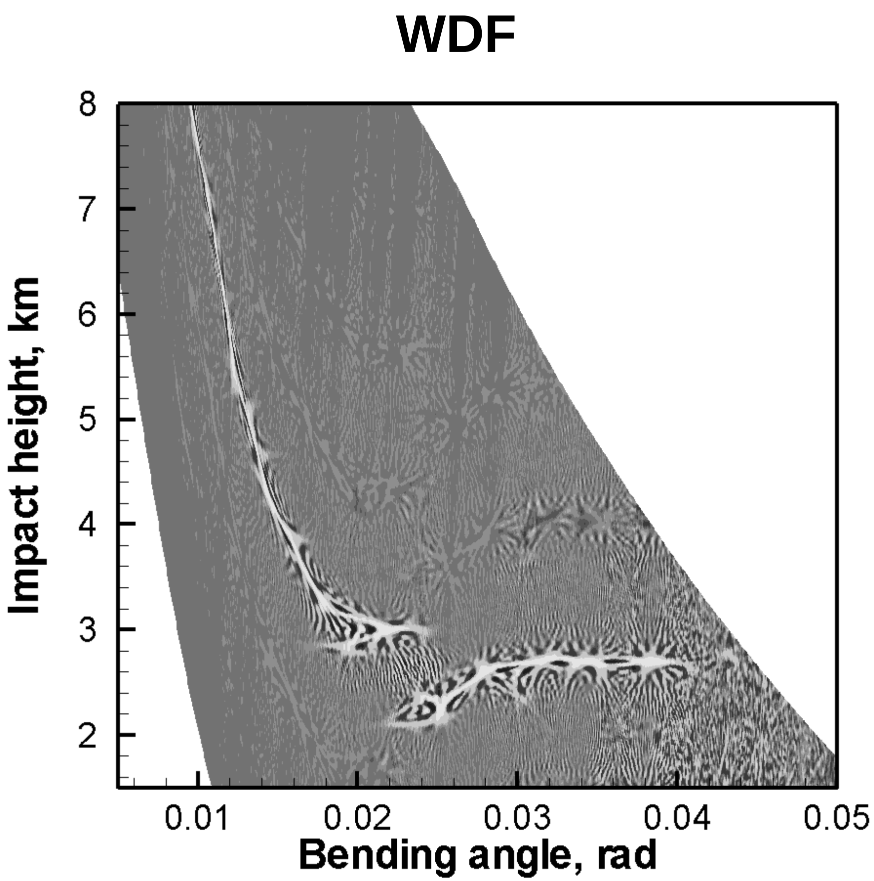
Dependence of WWDF from the window width. From left to right: increase of the window (4, 8, 16 s).

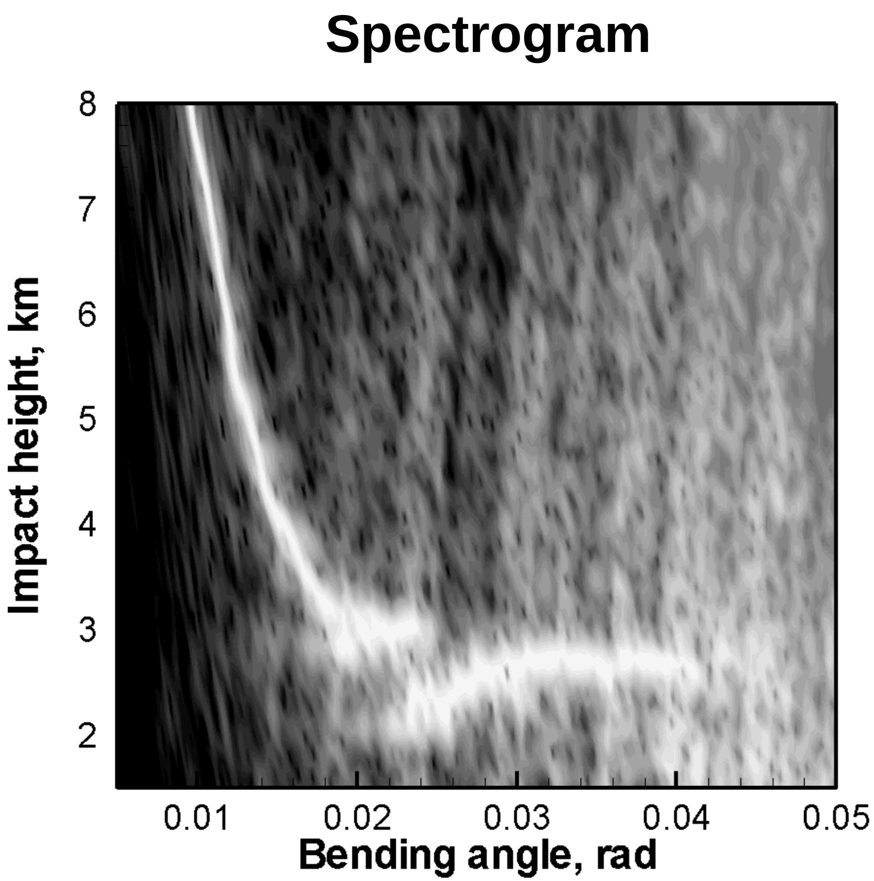
Processing artificial data



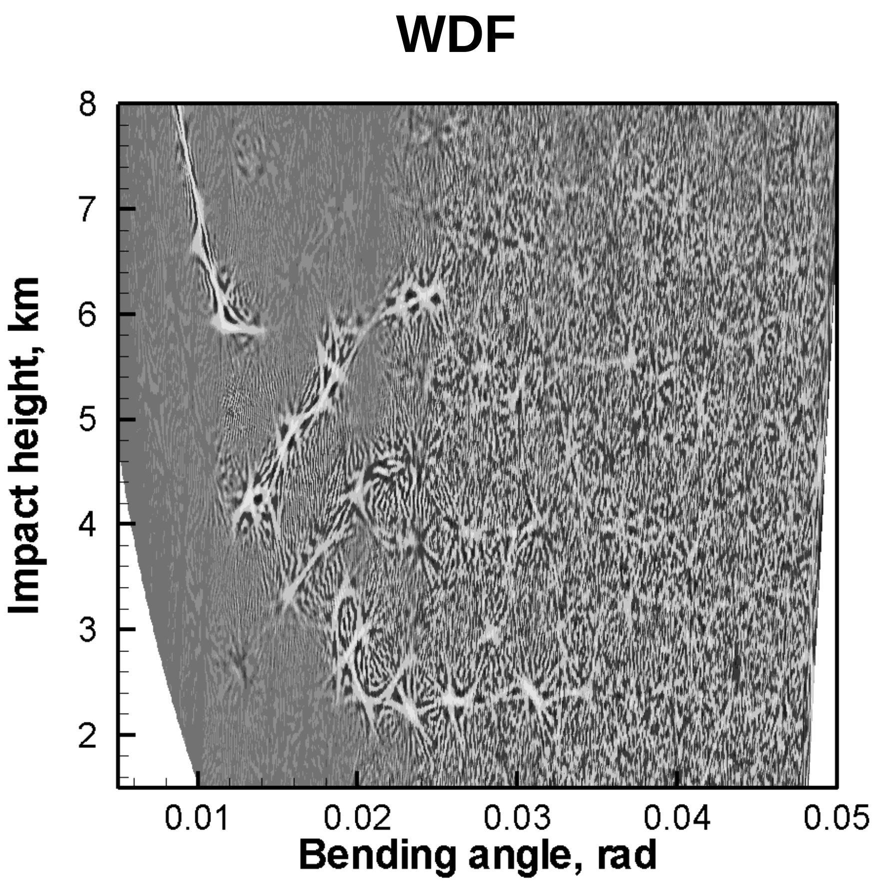
Left: artificial event without noise. **Right:** the same event with model of turbulent fluctuations superimposed (power spectrum, anisotropic turbulence).

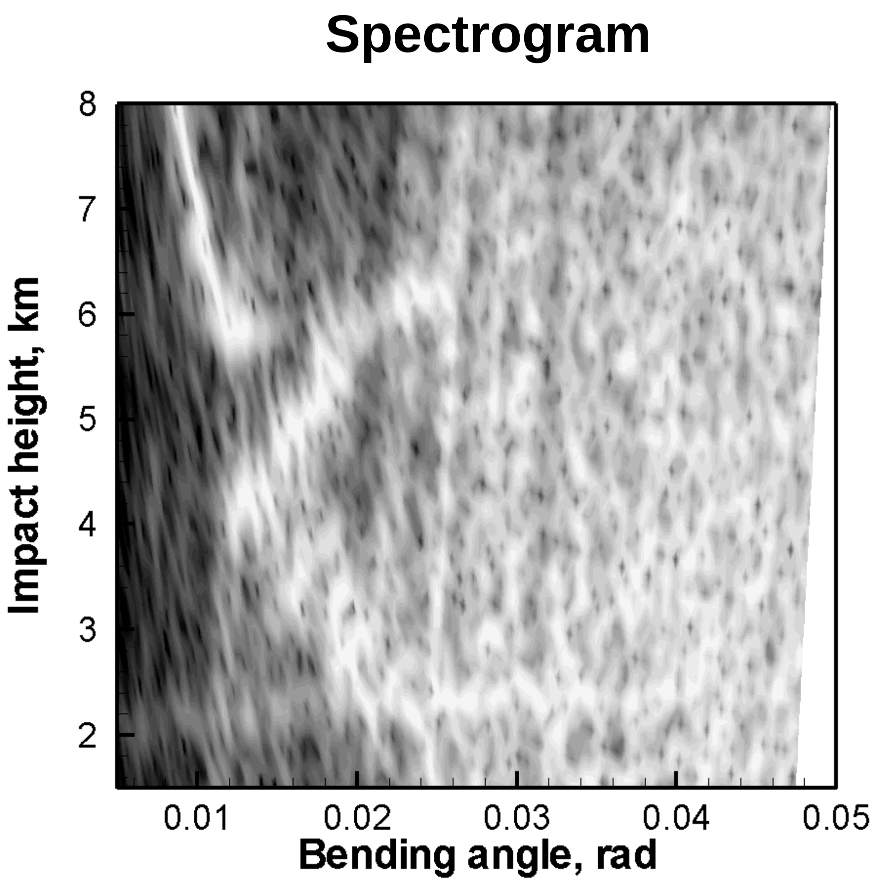
Processing COSMIC data





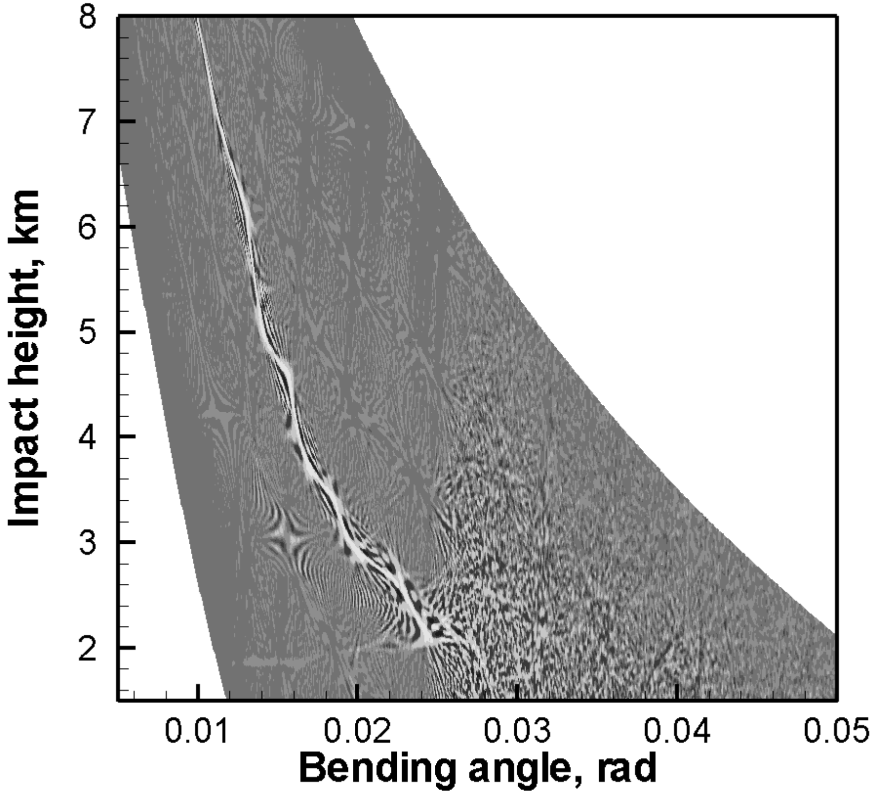
Processing COSMIC data



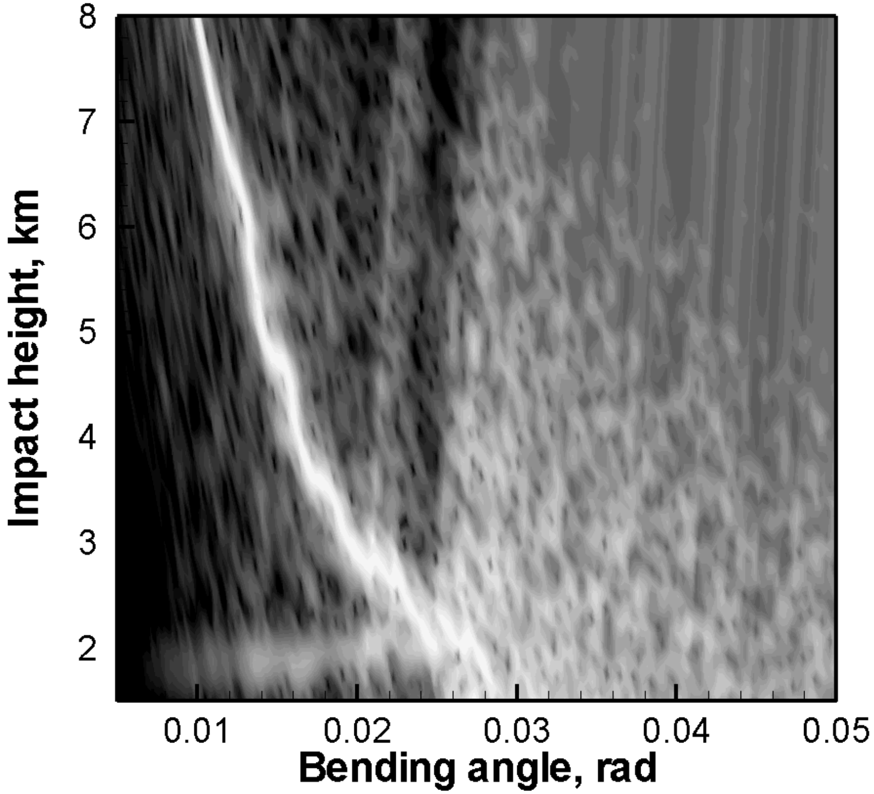


Processing COSMIC data

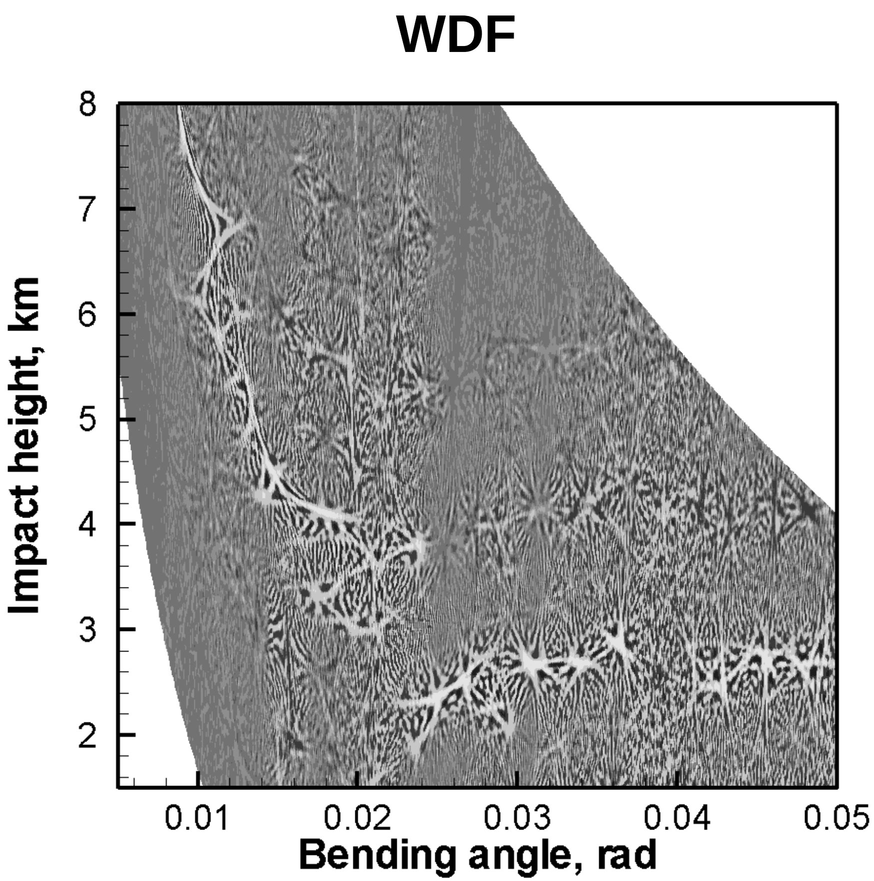
WDF

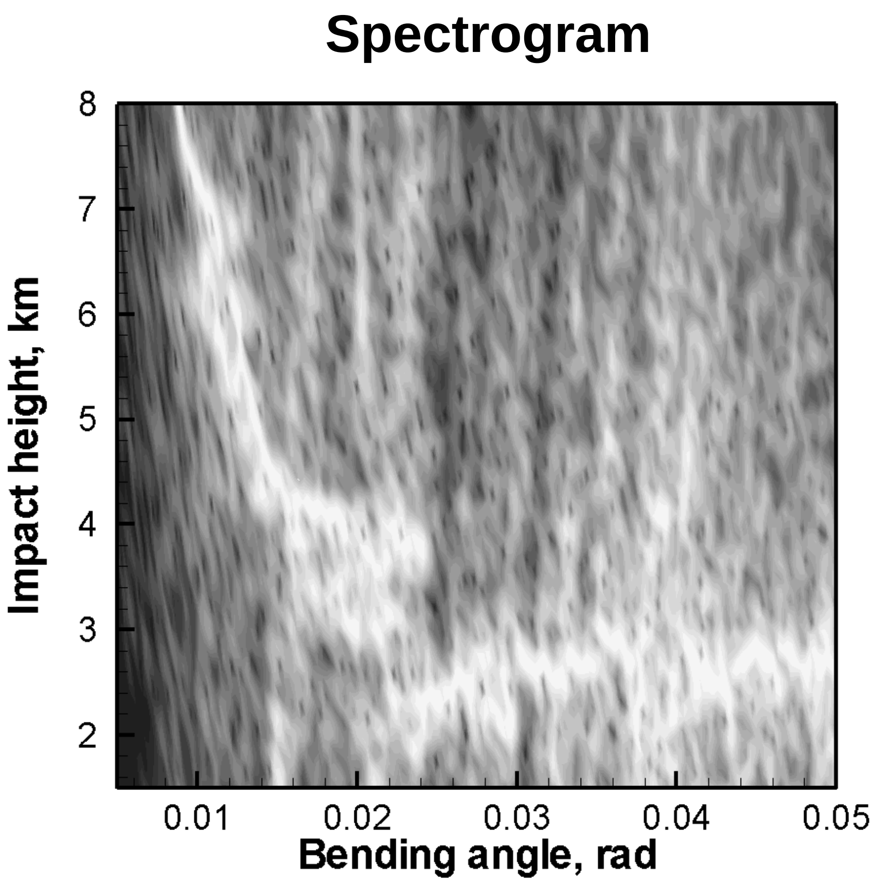


Spectrogram



Processing COSMIC data





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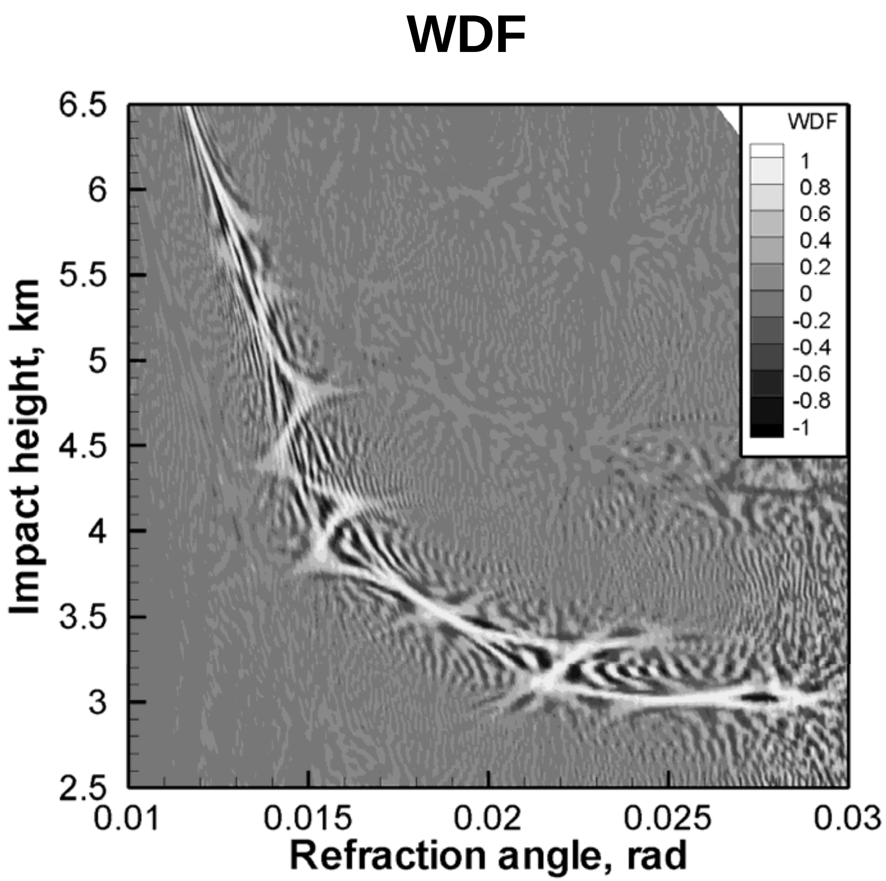
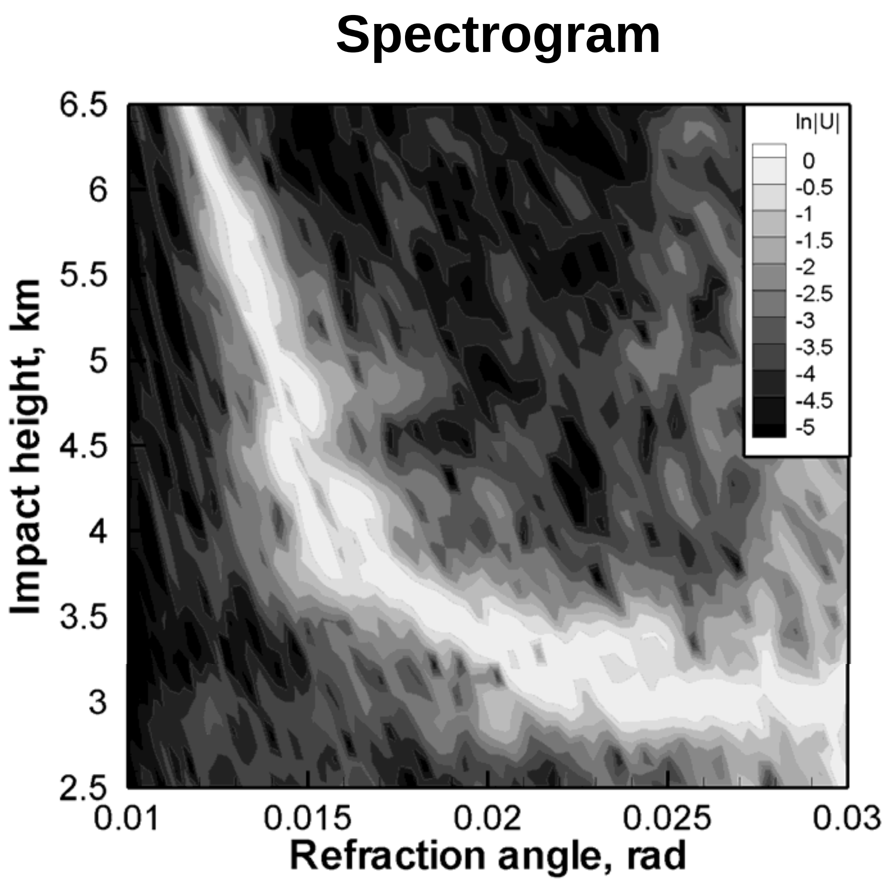
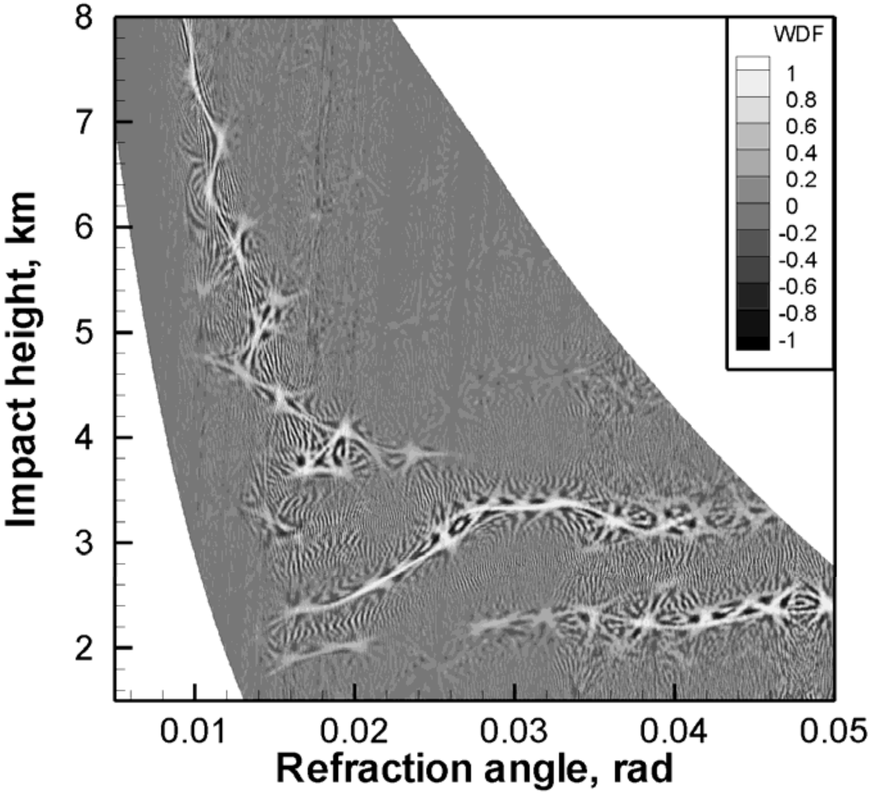


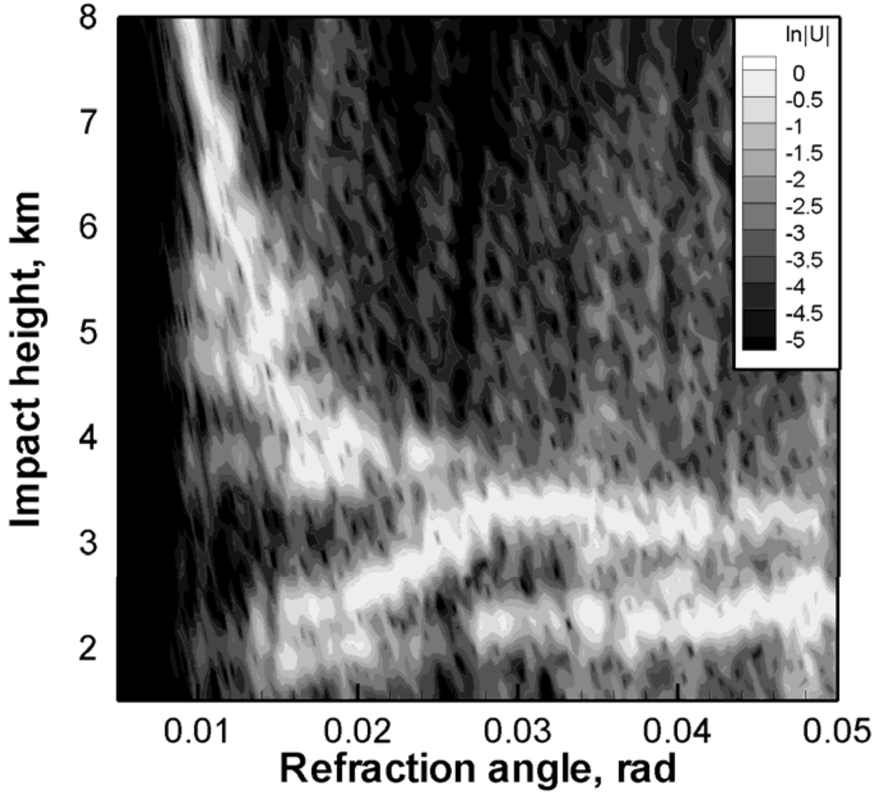


Processing COSMIC data

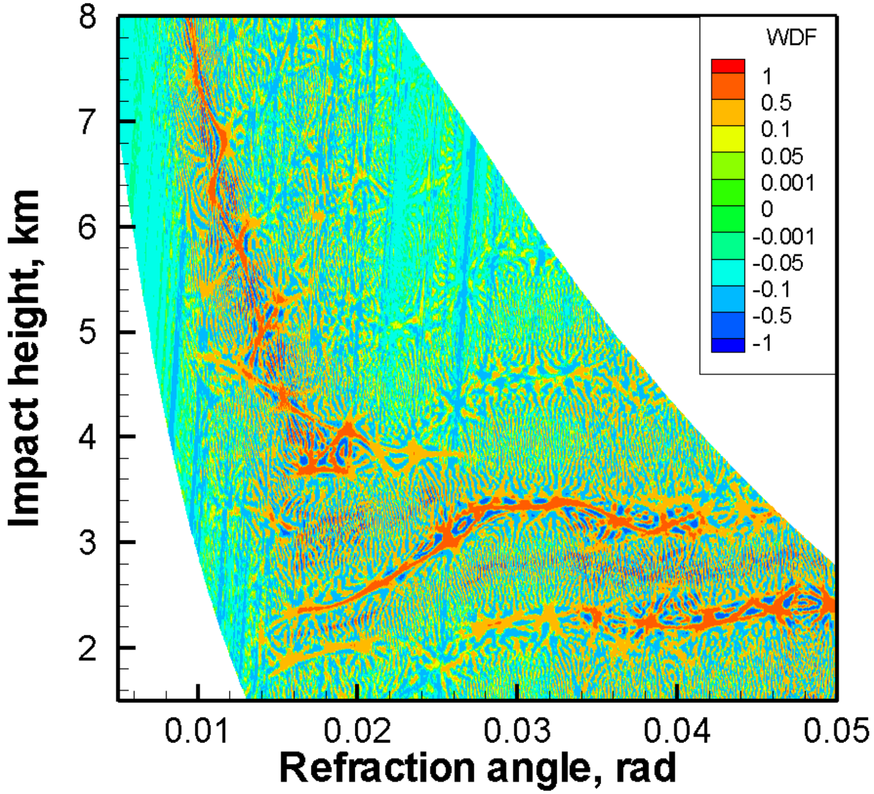
WDF

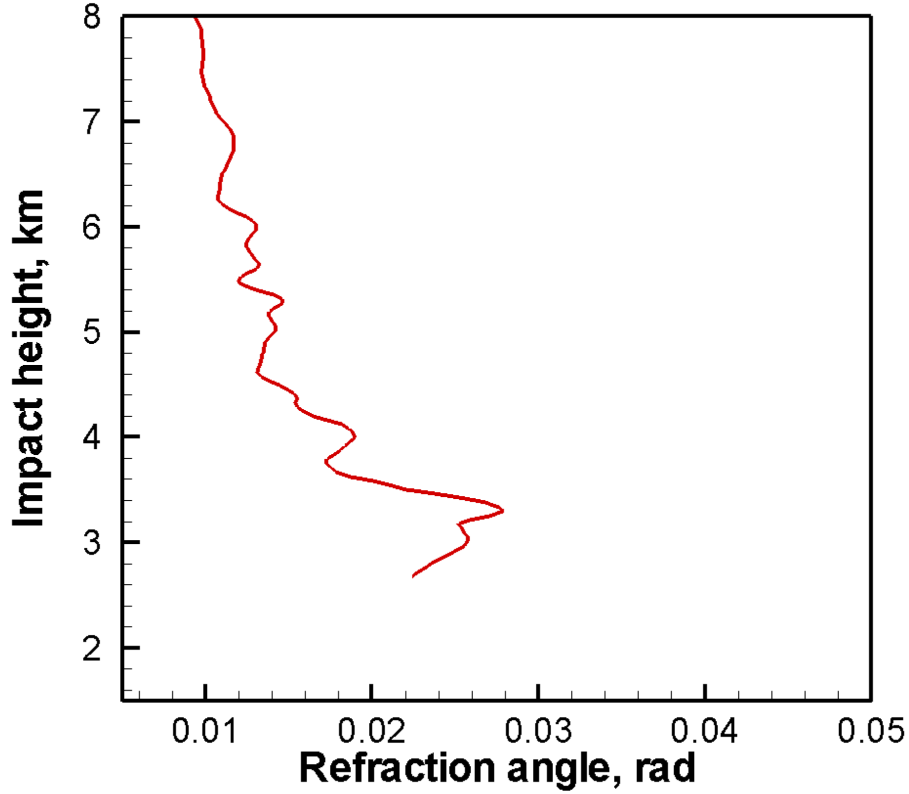


Spectrogram

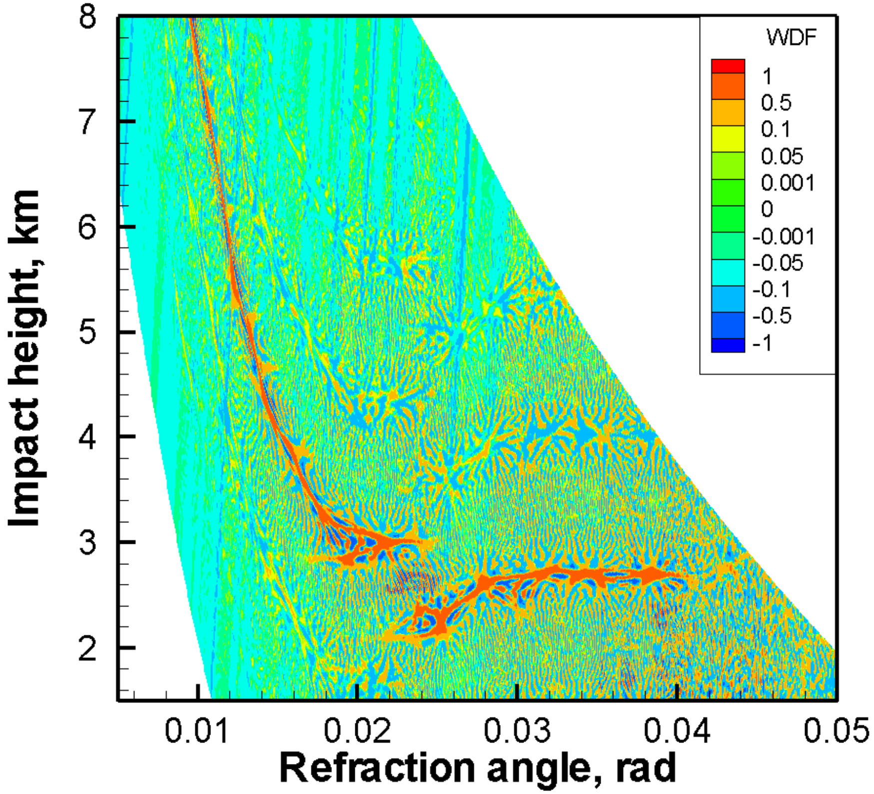


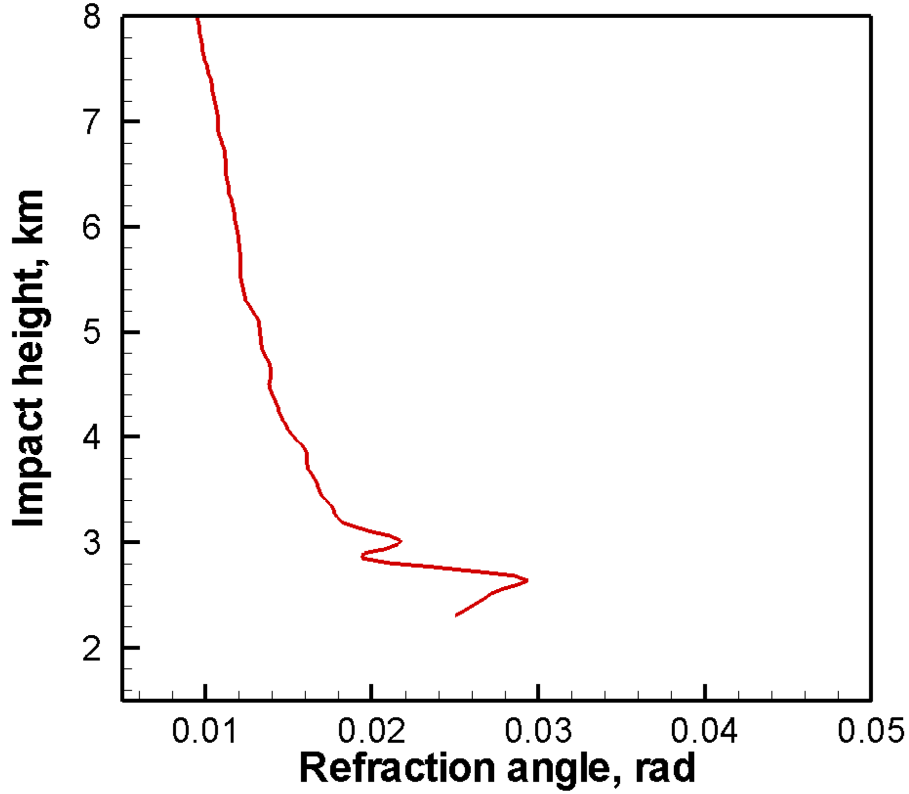
Negative bias



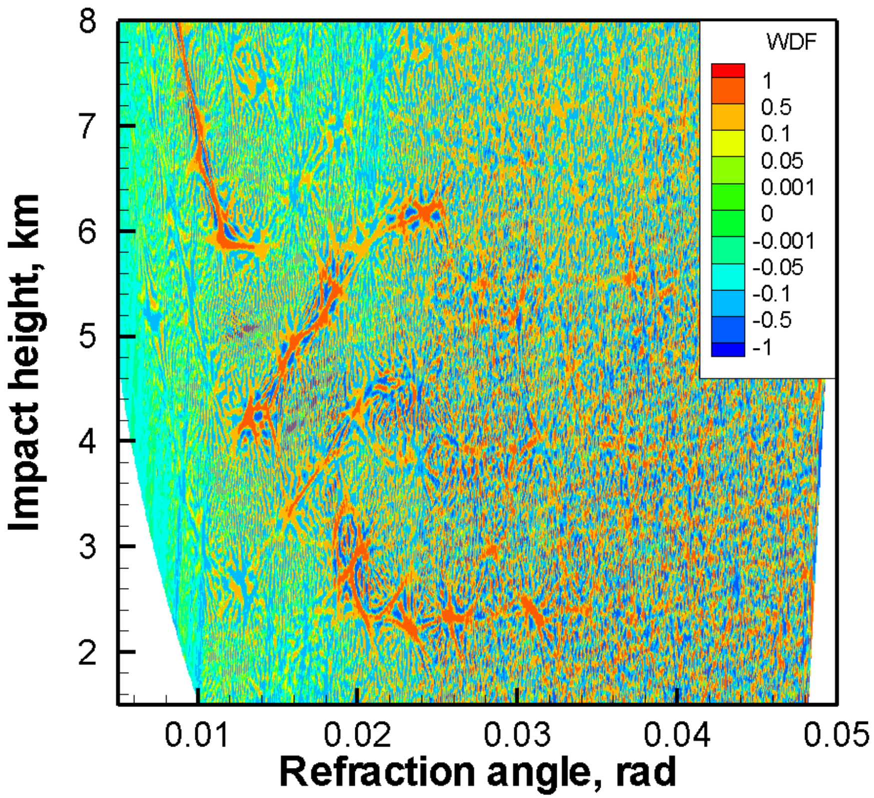


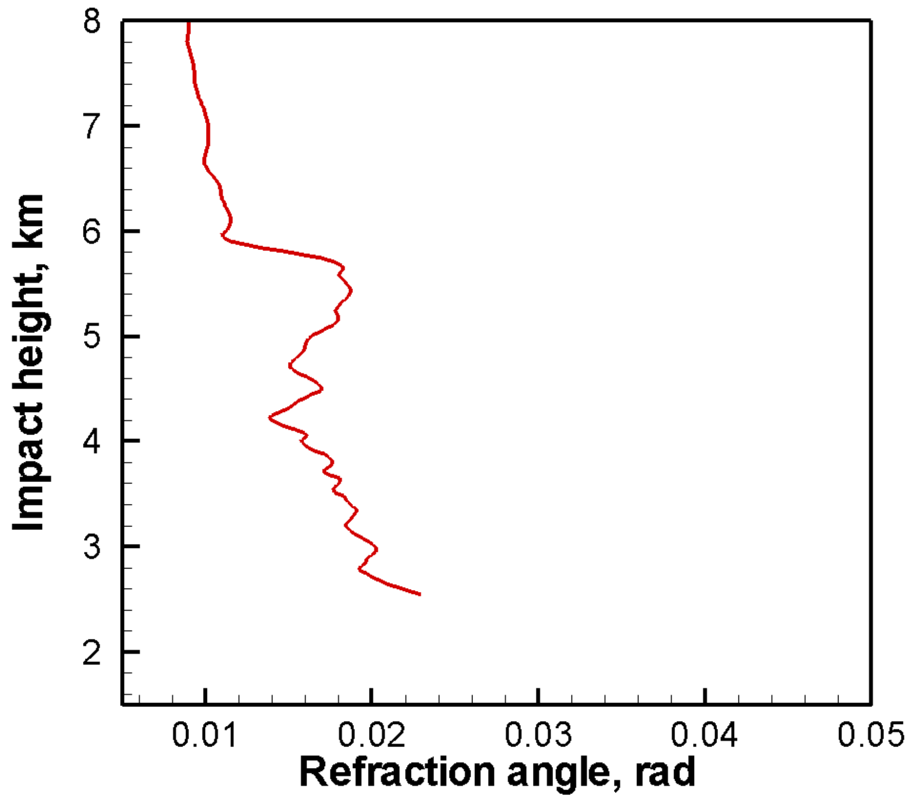
Negative bias





Negative bias





Conclusions

1. We suggested the application of Wigner Distribution Function for the retrieval of ray manifold structure from radio occultation measurements of wave fields.
2. Analysis of ray manifold structure in 2D phase space reveals that horizontal gradients may constitute a source of negative bias.
3. The only practical problem is what to do now.
4. We are going to consider the application of WDF for the retrieval of bending angles, ionospheric correction, noise filtering and the investigation of the structural uncertainty of radio occultation data.