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# Abel integral inversion in occultation measurements

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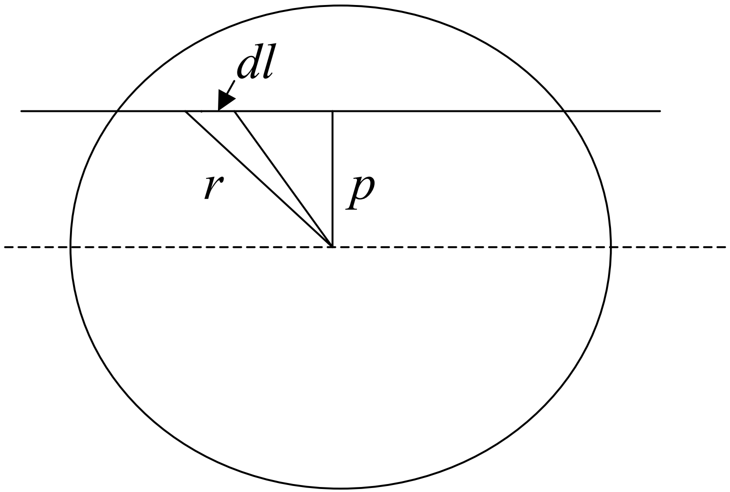
# Introduction




Occultation geometry

\+ Spherical symmetry

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Abel-type integral equation

$$u = \int_{(AB)} f(l)\,dl = 2\int_p^R \frac{f(r)r}{\sqrt{r^2 - p^2}}\,dr$$

$$dl = d\sqrt{r^2 - p^2} = \frac{r\,dr}{\sqrt{r^2 - p^2}}$$

| | Vertical inversion in stellar occultation | Reconstruction of atmospheric refractivity from refractive angle measurements |
|---|---|---|
| | horizontal column density $N(p)$<br>$\Downarrow$<br>local density $\rho(r)$ | Refractive angle $\alpha(p)$<br>$\Downarrow$<br>refractivity $\nu(r)$ |
| Forward model | $N(p) = V(\rho(r)) = 2\int_p^\infty \frac{\rho(r)\, r\, dr}{\sqrt{r^2 - p^2}}$ | $\alpha(p) = R(\nu(z)) = -2p\int_p^\infty \frac{\nu'(z)\, dz}{\sqrt{z^2 - p^2}}$ |
| Formal inversion | $\rho(r) = V^{-1}(N(p)) = -\frac{1}{\pi}\int_r^\infty \frac{N'(p)\, dp}{\sqrt{p^2 - r^2}}$ | $\nu(z) = R^{-1}(\alpha(p)) = \frac{1}{\pi}\int_z^\infty \frac{\alpha(p)\, dp}{\sqrt{p^2 - z^2}}$ |

Under what hypothesis on function and in what sense the Abel inversion is hold?

the integral is considered in Lebesque-Stiltjes sense

Is the problem well-posed?

# Well-posed problem

**Definition .** *Let X and Y be two linear normed spaces and let A: X→Y be a linear operator. The problem of solving the equation* $Ax=y$ *where* $y \in Y$ *is given and* $x \in X$ *is unknown , is well-posed if (and only if) the following conditions hold:*

- *(i) the equation has at least one solution*
- *(ii) the equation has at most one solution*

The functions under integral sign are continuous and of bounded variation

- *(iii) the solution x depends continuously on the right-hand side y*

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(iii): - $A^{-1}$ is continuous

- small change in data ⇒ small changes in solution

The problem is *ill- posed* if it is not well-posed

- the vertical inversion problem doesn't fulfill the condition *(iii)* ($V^{-1}(\rho)$ is not continuous) . It is ill-posed
- Refractive inverse problem is well-posed

Fractional integration operator

$$J^{\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{u(t)\,dt}{(x-t)^{1-\alpha}}$$

$$D^{\alpha}J^{\alpha}u(x) = u(x)$$

$$N = J^{\frac{1}{2}}(\rho)$$

$$\nu = J^{\frac{1}{2}}(\alpha)$$

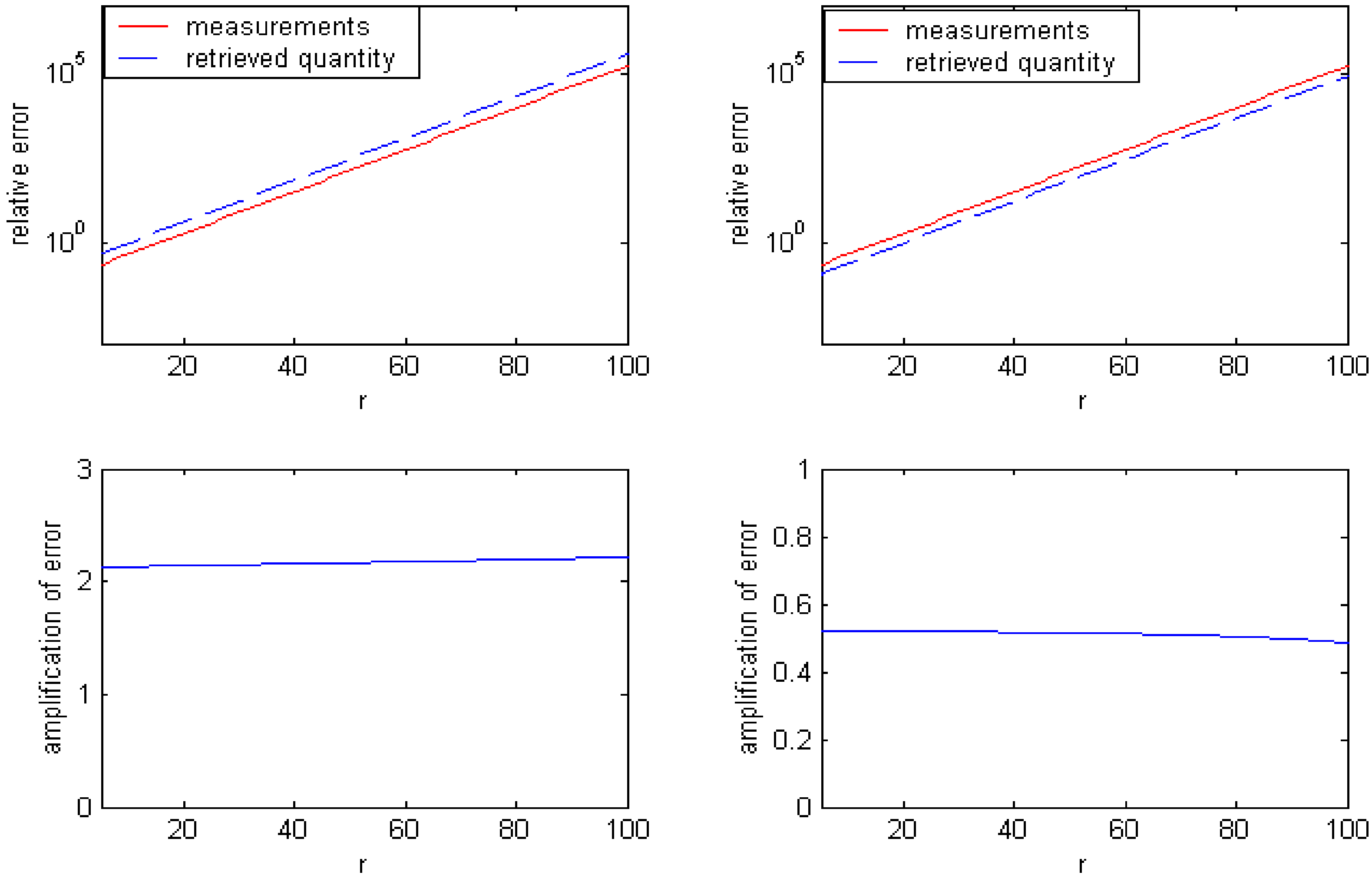
Amplification of error coefficient

$$a = \frac{\left|\frac{\delta x}{x}\right|}{\left|\frac{\delta y}{y}\right|}$$

$$\delta x_i^2 = C_x^{ii} \qquad \delta y_i^2 = C_y^{ii}$$

# Vertical inversion

# Refractive inversion



# Discretization

- Number of measurements is always finite
- Discretization makes the problem formally even-determined
- Comfortable upper limit 120 km
- occultation geometry : almost equidistant data grid in upper and middle atmosphere, becoming denser for lower altitudes
- standard methods of equidistant discretization should be applied
- weak singularity at the lower limit :
  - polynomial product integration methods (collocation)
  - estimation of the integral about lower limit
- testing of discretization schemes:
  - condition number, amplification of error
  - averaging kernel, Backus-Gilbert spread

$$s(z) = \int (z - z')^2 A^2(z, z') dz' \Big/ \left( \int A(z, z') dz' \right)^2$$

  - Modified Backus-Gilbert spread

$$s(z) = \int (z - z')^2 A^2(z, z') dz' \Big/ \int A^2(z, z') dz'$$

# Vertical inversion discretization

- Onion peeling (midpoint collocation method) $O(h^{3/2})$
- Polynomial onion peeling (second order polynomial collocation method) $O(h^3)$
- Trapezoidal rule with estimation of weak singularity at lower limit $O(h^{3/2})$

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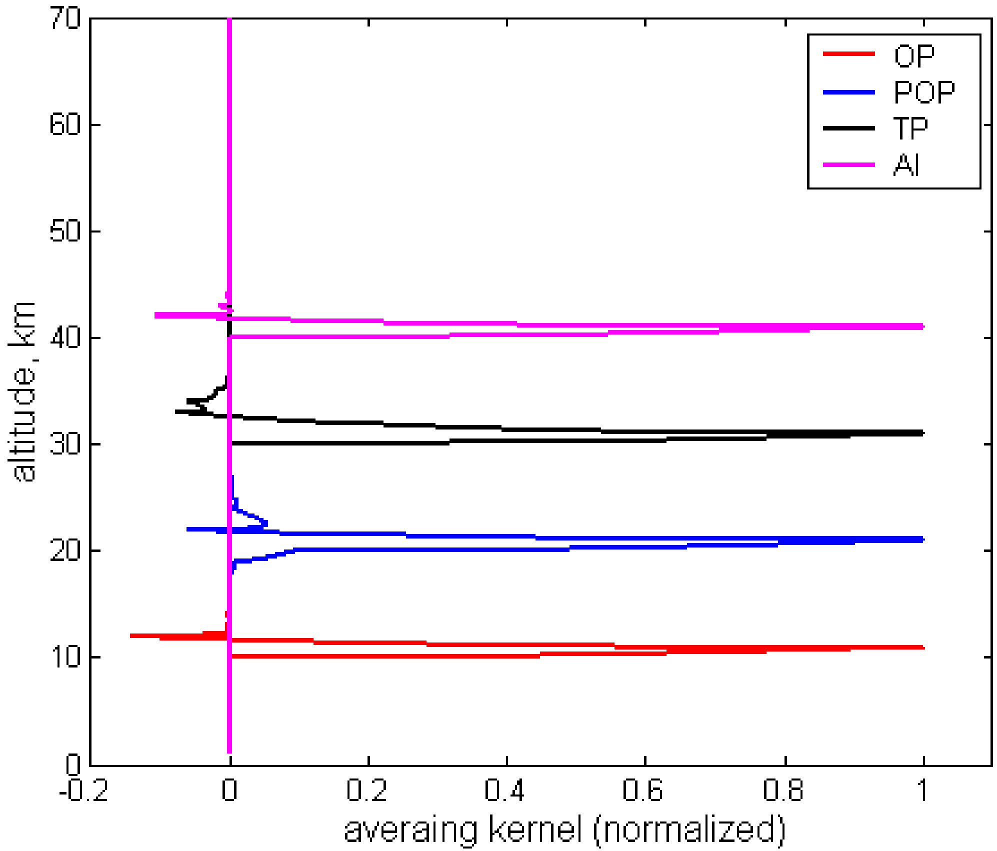
Triangular kernel matrix

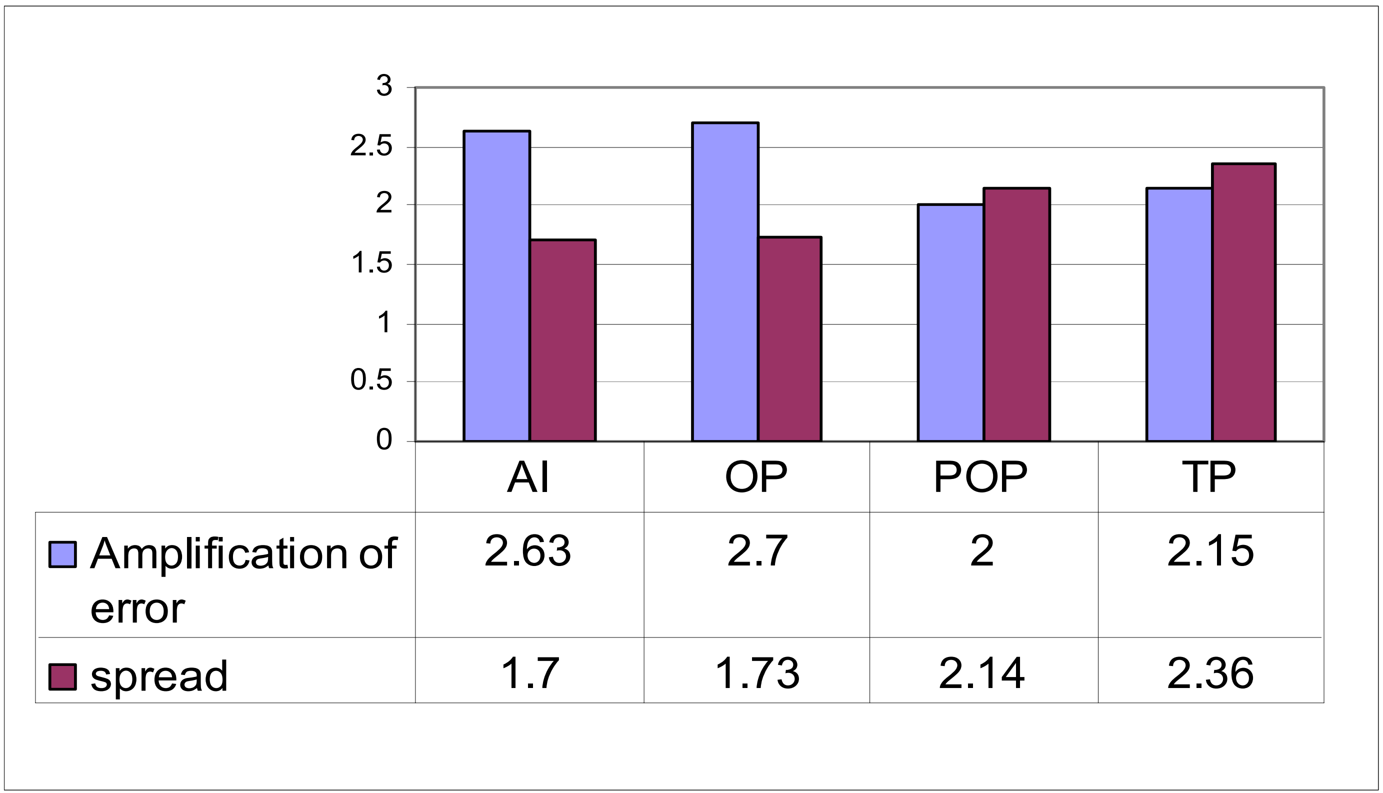
- Discretization of inverse Abel transform

# Discrete schemes for vertical inversion: smoothing properties




| | BGspread | spread |
|---|---|---|
| AI | 1.42 | 1.7 |
| OP | 1.52 | 1.73 |
| POP | 1.18 | 2.14 |
| TP | 1.66 | 2.36 |

| | AI | OP | POP | TP |
|---|---|---|---|---|
| Condition number | | 10.6 | 10.9 | 10.26 |
| Amplification of error | 2.63 | 2.7 | 2.0 | 2.15 |



# Discretization of refractive inversion

- Pole-free formulations
- 1-step: $\alpha = A \cdot \nu$ mid-point collocation

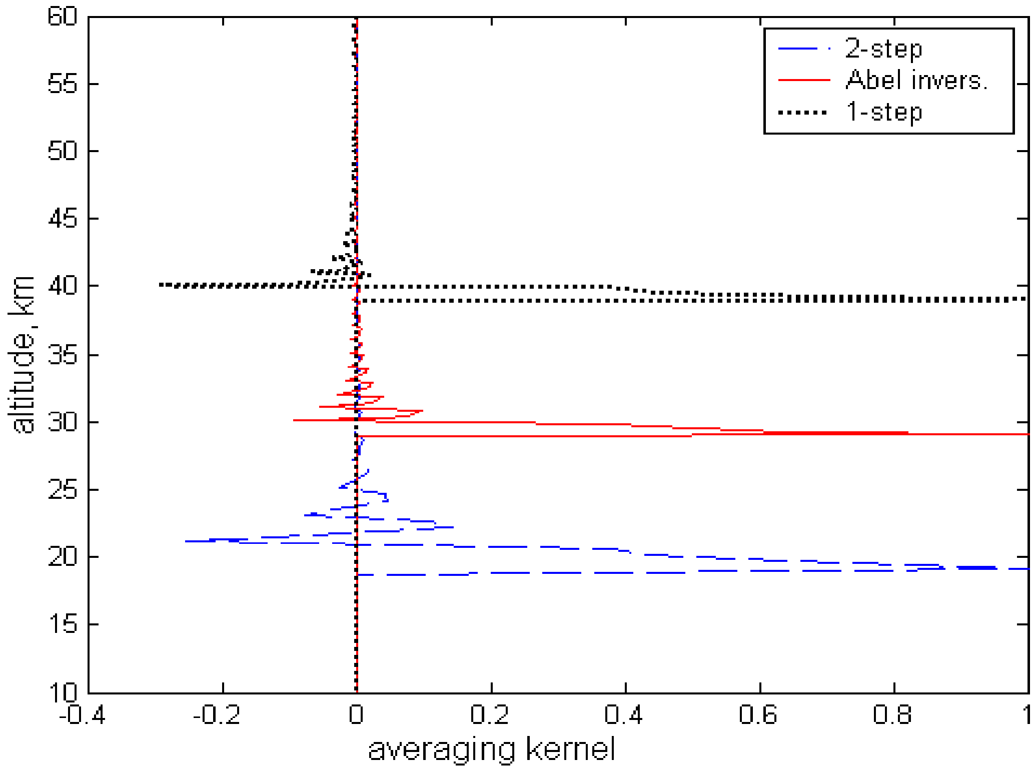
$$\alpha(p) = -2p \int_p^\infty \frac{\nu'(z)\,dz}{\sqrt{z^2 - p^2}}$$

$$\alpha(p_i) = -2p_i \left( \sum_{j=i}^{N-1} \frac{1}{\sqrt{r^2_{j+1/2} - r_i^2}} \int_{r_j}^{r_{j+1}} \nu' dr \right) = -2p_i \left( \sum_{j=i}^{N-1} \frac{\nu_{j+1} - \nu_j}{\sqrt{r^2_{j+1/2} - r_i^2}} \right)$$

- 2-step: $\dfrac{\alpha}{2p} = B \cdot \nabla \nu$ $\qquad \alpha \Rightarrow \nabla\nu \Rightarrow \nu$

- Discretization of inverse Abel transform

# Discrete schemes for refractive inversion




| | 1-step | 2-step | AI |
|---|---|---|---|
| Spread (km) | 2.84 | 10.7 | 2.97 |
| BG spread (km) | 4.45 | 12.6 | 4.55 |
| Amplification of error | 0.56 | 0.50 | 0.44 |

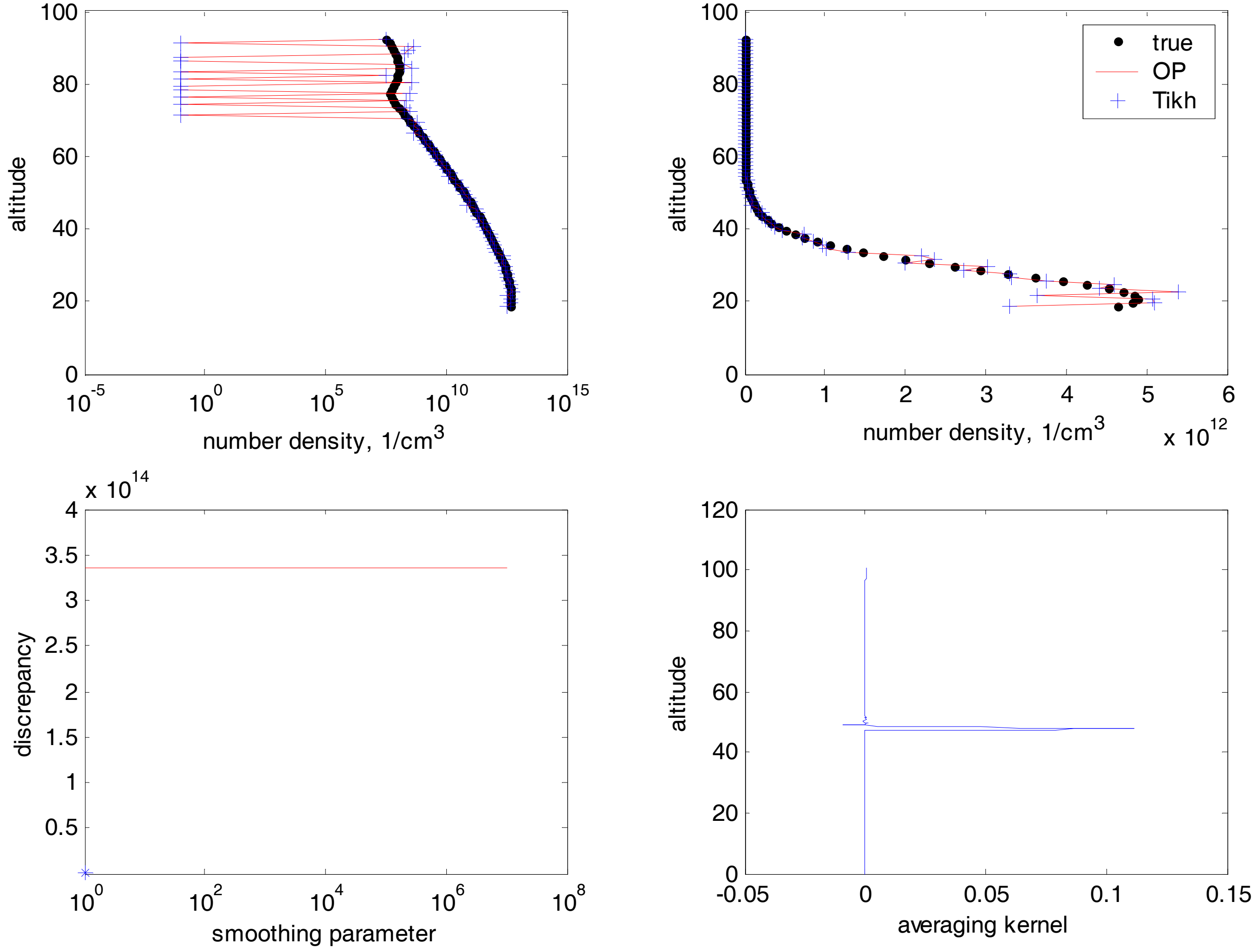
# Regularization

- Recommended for noisy data
- Trade-off: resolution and accuracy
- Tikhonov regularization for vertical inversion :

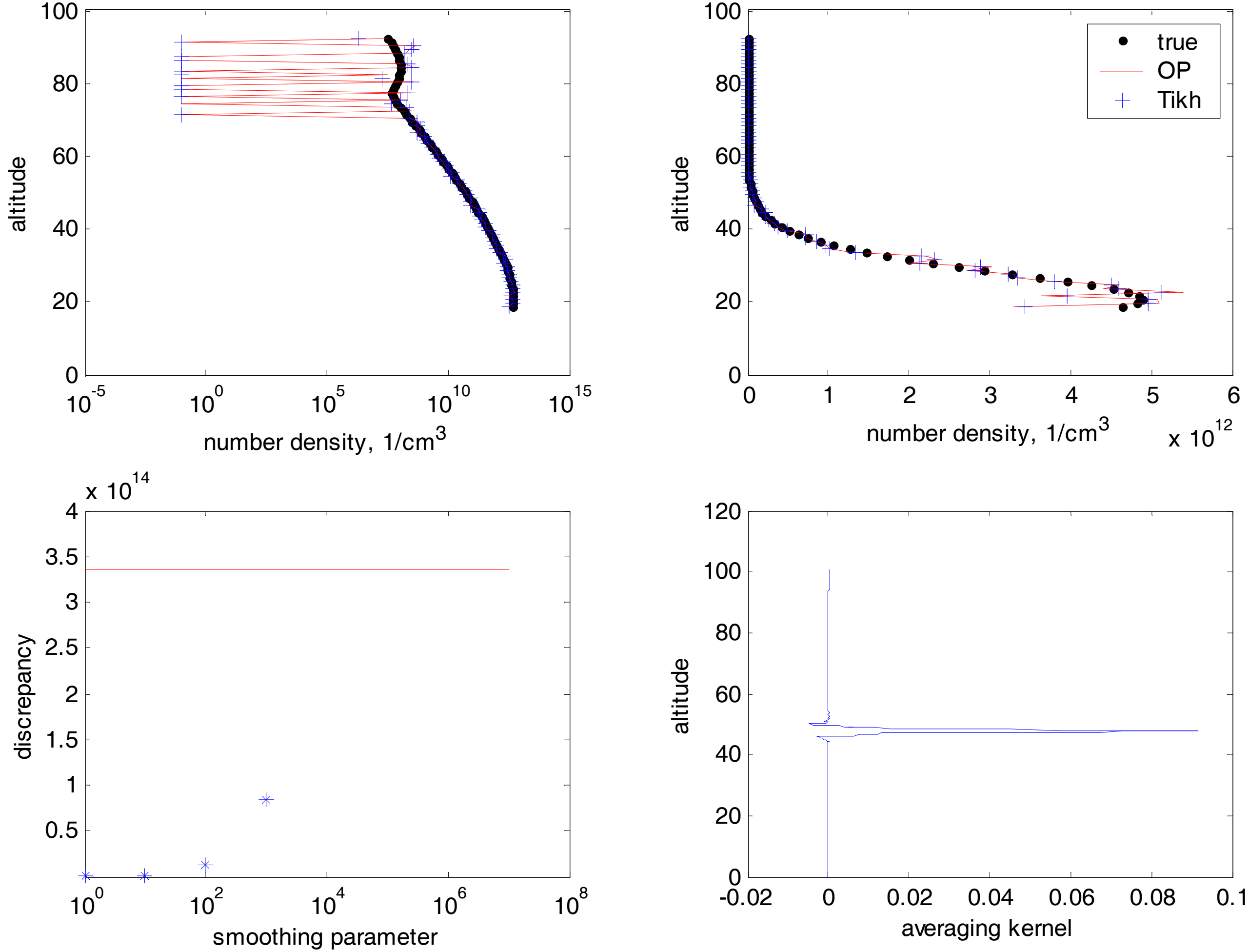
  Example:

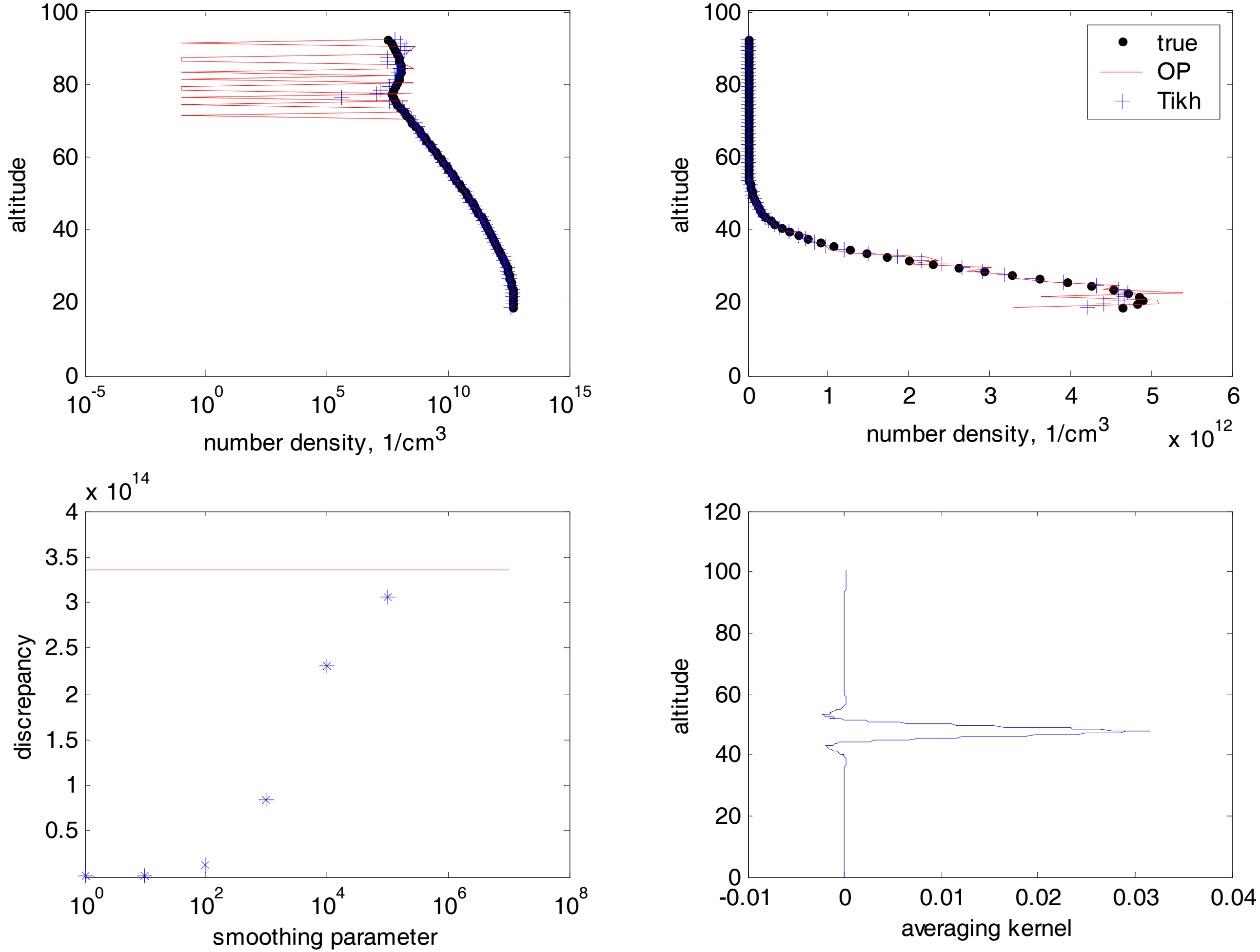
  Ozone profile reconstruction

  magnitude =4, T=10000 K

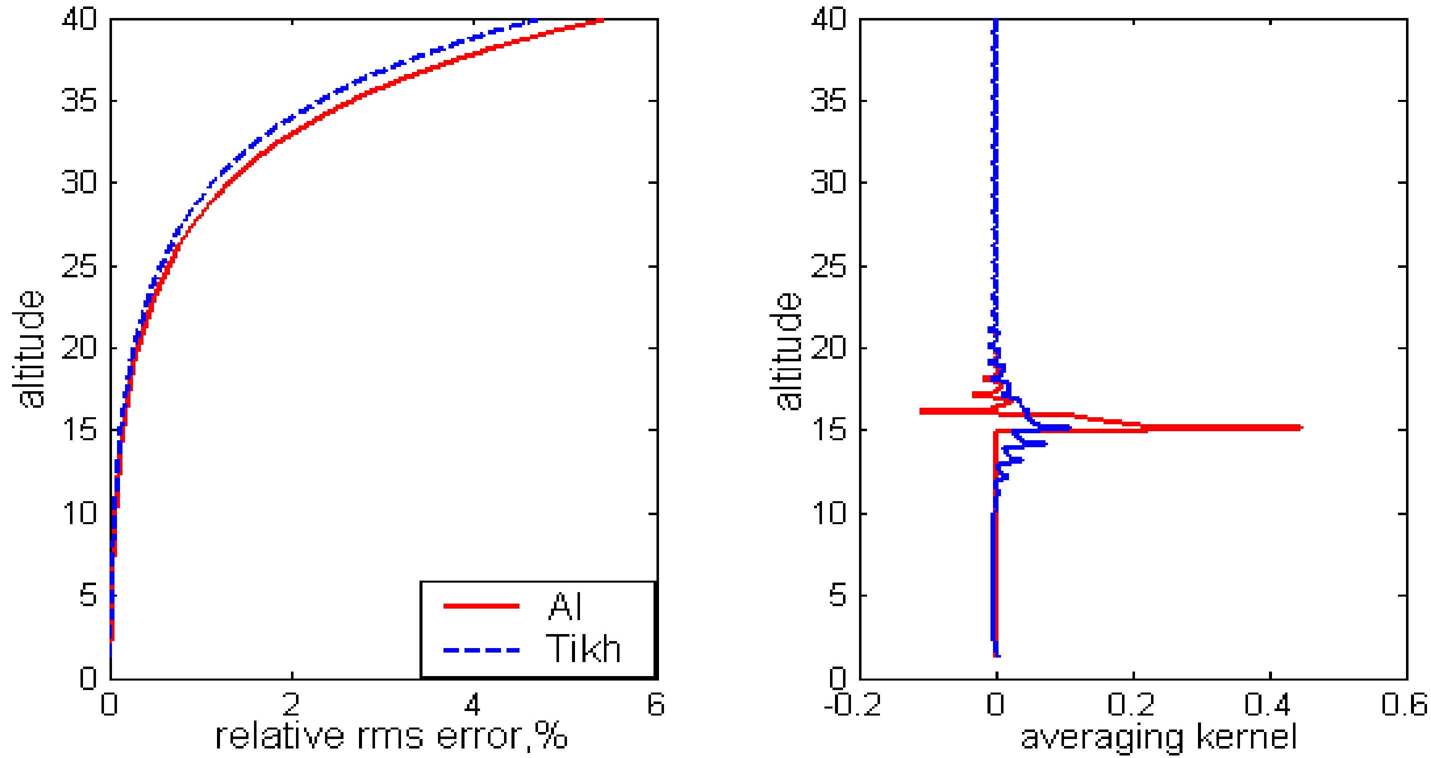


100
80
60
40
20
0
altitude
10^-5 10^0 10^5 10^10 10^15
number density, 1/cm^3
100
80
60
40
20
0
altitude
true
OP
Tikh
0 1 2 3 4 5 6
number density, 1/cm^3
x 10^12
x 10^14
4
3.5
3
2.5
2
1.5
1
0.5
discrepancy
10^0 10^2 10^4 10^6 10^8
smoothing parameter
120
100
80
60
40
20
0
altitude
-0.05 0 0.05 0.1 0.15
averaging kernel


altitude
number density, 1/cm3
true
OP
Tikh
x 10^12
discrepancy
x 10^14
smoothing parameter
averaging kernel


altitude
number density, 1/cm3
true
OP
Tikh
x 10^12
discrepancy
x 10^14
smoothing parameter
averaging kernel

# Regularization : refractive inversion




- some improvement is observed
- Test case:
  - 1 km layer structure
  - Noise: uniform for all altitudes; noise/refractive angle (surface)=$3*10^{-4}$

# Summary

- Two Abel-inversion problems are considered
  - Ill-posed vertical inversion problem : amplification of error >1
  - Well-posed refractivity reconstruction problem : amplification of error <1
- Efficiency of discretization: schemes with smaller amplification of error have worse resolution
- Vertical inversion : in-pair comparison of discretization schemes
  - Discretized inverse Abel transform is better than onion peeling
  - Quadratic polynomial replace is better than trapezoidal rule in pole formulation
- Discretization of refractive inversion: inverse Abel transform shows the best results
- Regularization of Tikhonov type.
  - Accuracy of retrieval is improved, resolution is degraded
  - Significant improvement of reconstruction accuracy for vertical inversion
  - Some positive effect also for refractivity inversion