Canonical Transform Methods for Radio Occultation Data

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Summary. In the present work, we study a canonical transform that directly maps the measured field to the impact parameter representation without first carrying out a back-propagation. This canonical transform is determined to first order in a small parameter that measures the deviation of the satellite orbit from a circle. When the parameter is equal to zero, i.e., for circular orbits, then our canonical transform simply reduces to a Fourier transform. In the general case, the form of the generating function is such that it does not directly allow an implementation as an FFT-like algorithm. We are currently investigating the possibility of decomposing the mapping in order to obtain a fast, efficient numerical implementation.

1 Introduction

The idea of transforming radio occultation measurements from the space representation to the impact parameter representation was recently shown to be a powerful retrieval method [1]. This tranformation is given by a specific canonical transform whose generating function dictates the form of the Fourier integral operator performing the mapping of the field to the ray coordinate representation. It turns out that back-propagation is an immanent part of the canonical transform. Simulations performed with global atmospheric fields for cases with strong water vapor gradients that give rise to multipath propagation show that the canonical transform method can unfold the multipath behavior, thereby leading to more accurate retrieved quantities.

2 Multipath

In the present paper we investigate canonical transform (CT) retrieval methods for radio occultations. The geometry $\mathbf{r} = (x, y)$ of the system is chosen such that the x-axis coincides with the initial propagation direction of the radio waves (see Fig. 1). For simplicity we can assume the GPS satellite to

be located infinitely far away. The origin of the coordinate system is located in the center of curvature of the occultation point. The canonical momentum associated to y is denoted η. The signal is measured along the low-Earth orbit (LEO) trajectory and we will also refer to this trajectory as a time direction. The impact parameter is denoted p and the bending angle ϵ.
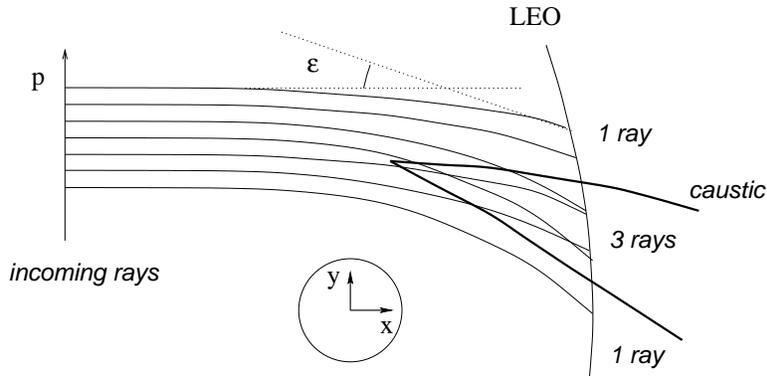

Fig. 1. Multipath example: water vapor layers in the lower troposphere give rise to the formation of a caustic (in the example it is a so-called cusp caustic) which surrounds a region with 3 rays passing through any point, implying that 3 rays are arriving at the same time instant at the LEO orbit.

The field measured at the LEO satellite is a solution $u(\mathbf{r}) = A(\mathbf{r})\exp(i\phi(\mathbf{r}))$ to the Helmholtz equation,

$$\left(\nabla^2 + k^2 n^2(\mathbf{r})\right) u(\mathbf{r}) = 0 \,, \tag{1}$$

where $k = 2\pi/\lambda$ is the wave-vector of the radio wave emitted from the GPS satellite. The atmosphere is characterized by an index of refraction $n(\mathbf{r})$. In the geometric optics limit $\lambda \rightarrow 0$ we can use the WKB (Wentzel-Kramers-Brillouin) ansatz to arrive at the solution

$$u(\mathbf{r}) = \frac{1}{\sqrt{j(\mathbf{r})}} \exp\left(ikS(\mathbf{r})\right) \,. \tag{2}$$

The quantity $S(r)$ is the eikonal and $j(\mathbf{r})$ is the Jacobian of the transformation from (x, y) to ray coordinates and it measures the divergence of a pencil of rays. The points where $j(\mathbf{r}) = 0$, i.e., where the rays tend to focus, define the caustics of the field $u(\mathbf{r})$ (see Figs. 1 and 2) [2].

Since the wavelength is much smaller than typical atmospheric variations, the WKB solution is expected to be a good approximation to the true solution. However, water vapor layers in the troposphere can lead to multipath (MP) propagation implying that the field $u(\mathbf{r})$ will contain caustics. As a result, the WKB solution will breakdown within a region surrounding the caustics.

There exists various methods (e.g., uniform approximations, Maslov theory, and Fourier integral operators) to improve the WKB solutions in multipath regions [3].
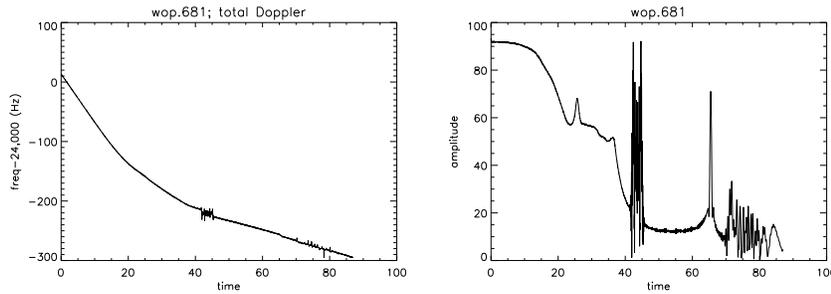

Fig. 2. Simulated example (with a wave optics propagator) showing the Doppler (derivative of the phase) and amplitude of a radio signal at the LEO orbit. One observes multipath (MP) behavior with 3 interfering rays appearing around the time 42-46 seconds.

3 Ray Manifold

Figure 3 schematically shows the propagation of rays from the GPS satellite towards the LEO satellite in phase space. Initially, the impact parameter coordinate corresponds to the y coordinate whereas along the LEO orbit one has approximately that the impact parameter is proportional to η.
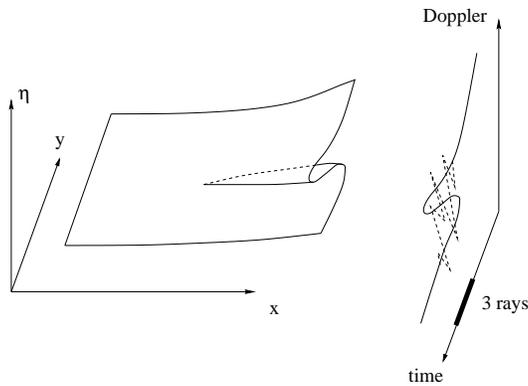

Fig. 3. Schematic drawing of phase space showing the ray manifold and its projection onto the Doppler vs. time plot. The MP region results in an irregular Doppler signal.

4 Canonical Transform with Back-Propagation

We are interested in calculating the bending of individual rays. To this end, one can carry out a transformation of the Helmholtz equation in vacuum to a new set of coordinates that will allow us to extract the ray behavior in multipath regions. The transformation of phase space will be given by a canonical tranformation that maps from (y, η) to (z, ξ), where z is the new coordinate and ξ its associated canonical momentum [4]. The natural choice of z is the impact parameter p because rays are uniquely defined by their impact parameters for a spherical symmetric atmosphere [1]. This means that MP behavior is unfolded in the p-representation (cf. Fig. 4).
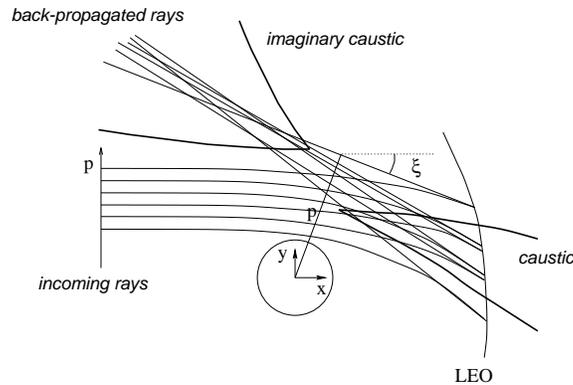
Fig. 4. Schematic drawing of back-propagation and (p, ξ)-representation.

Next, we use Egorov's theorem from the theory of Fourier integral operators (FIO) which yields an operator Φ_1 that will transform the field u from the (y, η) representation to the (z, ξ) representation [5]. Egorov's theorem states that this mapping is given by the operator

$$\Phi_1 u(z) = \frac{k}{2\pi} \int a_1(z, \eta) e^{ikS_1(z,\eta)} \widetilde{u}_x(\eta) d\eta \,. \tag{3}$$

Here, $a_1(z, \eta)$ is the symbol of the operator and the phase function $S_1(z, \eta)$ is the generating function of the canonical transform discussed below. The field $\widetilde{u}_x(\eta)$ is the Fourier transform of the back-propagated signal $u_x(y)$.

The Fourier transform (FT) is defined by the following equation:

$$Ff(\eta) = \widetilde{f}(\eta) = \int e^{-iky\eta} f(y) dy \,, \tag{4}$$

and the inverse Fourier transform reads

$$F^{-1}\widetilde{f}(y) = f(y) = \frac{k}{2\pi} \int e^{iky\eta}\widetilde{f}(\eta)d\eta \, . \tag{5}$$

It can be noted that the generating function $S_1(z, \eta) = z\eta$ yields a canonical transform which is the identity, and the associated Fourier integral operator is also the identity. The generating function $S(z, \eta) = x\sqrt{1 - \eta^2} + z\eta$ can be shown to give rise to a Fourier integral operator which corresponds to the back-propagation method [1]. The canonical transform recently investigated in [1] is characterized by the generating function

$$S_1(z, \eta) = z \arcsin \eta - x\sqrt{1 - \eta^2}. \tag{6}$$

The new phase space coordinates read

$$z = -x\eta + y\sqrt{1 - \eta^2} \, , \tag{7}$$

$$\xi = \arcsin \eta \, , \tag{8}$$

which means that the new coordinate z is the impact parameter p of the ray from the GPS satellite, and ξ is the ray direction angle with respect to the x-axis, i.e., $-\xi$ is simply the bending angle ϵ (note, when the GPS satellite is located at a finite distance a correction term has to be included).

The transformed wave function $\Phi_1 u(p)$ has the WKB form

$$\Phi_1 u(p) \approx A(p) \exp\left(ik \int_{p_0}^{p} \xi(p')dp'\right) , \tag{9}$$

where the canonical momentum can be obtained as follows:

$$\xi(p) = \frac{1}{k}\frac{d}{dp} \arg \Phi_1 u(p) \, . \tag{10}$$

5 Setting the Scene

We will now discuss the geometry shown in Fig. 5. The equation for the impact parameter reads:

$$p = r \cos \gamma = r \cos(\alpha + \beta) = r\eta \cos \beta + r\sqrt{1 - \eta^2} \sin \beta, \tag{11}$$

where the wave vector along the LEO orbit is given by $\eta = \cos \alpha$. Note, in the following y denotes the arc length coordinate along the LEO orbit and η is the associated canonical momentum.

Note, one can obtain the following estimates of relevant physical quantities:

- analyses of GPS/MET orbits show that β is of the order of $10^{-4} - 10^{-3}$ radians with a variation of about 10^{-4} during an occultation.

- the distance r to the LEO orbit varies by about 100 meters during an occultations.
- the order of MP effects on the (total) Doppler can be estimated to be about 3 Hz (corresponding to a deviations of $\frac{1}{3} \times 10^{-3}$ radians of MP rays).
- the effect of non-circular orbits results in a shift of the Doppler frequency of the order of 1 Hz.
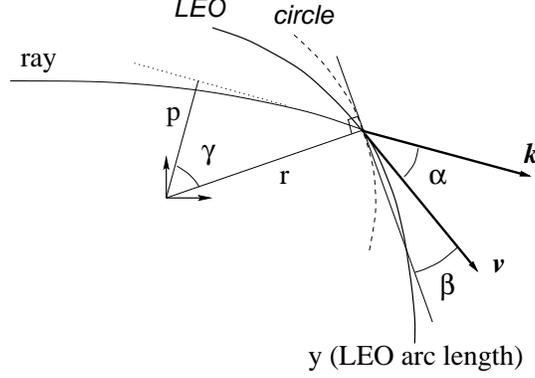

Fig. 5. Definition of the angles α, β and γ. Note, along the LEO orbit arc length y we have $\eta = \cos\alpha$.

6 Canonical Transform without Back-Propagation

In order to proceed, we will assume that $\beta = $ const (which is much smaller than unity). This implies that we take $r(y) = r_0 + \beta(y - y_0)$, where y_0 is some reference point which can be chosen in the middle of the MP region. In the following we will take $y_0 = 0$.

The structure of our CT approach is as follows:

- Perform a CT using the field $u(y)$ along the LEO orbit: transform from (y, η) to (z, ξ)
- FT corresponds to the following: F: $z = \eta, \xi = -y$; F^{-1}: $z = -\eta, \xi = y$;
- Egorov's FIO (of type 1):

$$\Phi_1 u(z) = \frac{k}{2\pi} \int a_1(z, \eta) e^{ikS_1(z,\eta)} \widetilde{u_x}(\eta) d\eta \,. \tag{12}$$

The associated generating function is:

$$dS_1 = \xi dz + y d\eta \,. \tag{13}$$

- Compose F^{-1} and Φ_1 and obtain a FIO (of type 2):

$$\Phi_2 u(z) = \int a_2(z,y) e^{ikS_2(z,y)} u(y) dy, \tag{14}$$

  with generating function

$$dS_2 = \xi dz - \eta dy. \tag{15}$$

- For the mapping F it follows that $dS_2 = -d(zy)$, i.e., here Φ_2 reduces to a FT: $\Phi_2 u \equiv F u$. Thus, one can think of Φ_2 as a "deformed" FT.

Thus, we seek a CT that maps from (y, η) to (z, ξ) with a generating function $S_2 = S_2(z, y)$, where

$$z = p = p(y, \eta), \tag{16}$$

given in Eq. (11). Here, $\xi(y, \eta)$ is to be determined. Then, the transformed wave function $\Phi_2 u(z)$ will unfold MP. It will have the WKB form

$$\Phi_2 u(p) \approx A(p) \exp\left(ik \int_{p_0}^{p} \xi(p') dp' \right), \tag{17}$$

where $\xi = \xi(p)$ follows from the derivative of the phase. Using $y = y(z, \xi)$, one obtains $y(z)$ (and $r(y)$, $\beta(y)$) and this yields the bending angle $\epsilon(p)$.

7 Generating Function and New Momentum

Using the canonical formalism (see, e.g., [4]) we obtain the following results (with $y_0 \equiv 0$):

$$z = r_0 \eta + y \eta \beta - r_0 \sqrt{1 - \eta^2}\, \beta + O(\beta^2), \tag{18}$$

$$\xi = -\frac{y}{r_0} + \frac{1}{2}\frac{y^2}{r_0^2}\beta - \beta + \frac{y}{r_0}\frac{\eta}{\sqrt{1 - \eta^2}}\beta + O(\beta^2). \tag{19}$$

The generating function reads:

$$S_2(z, y) = -\frac{zy}{r_0} + \frac{1}{2}\frac{zy}{r_0^2}\beta - y\sqrt{1 - z^2/r_0^2}\, \beta - z\beta + O(\beta^2). \tag{20}$$

Note the case of circular orbits ($\beta = 0$): This implies, $z = r_0 \eta$, i.e., the impact parameter is proportional to the Doppler frequency. In addition, $\xi = -y/r_0$, and $S_2 = -zy/r_0$, i.e., Φ_2 reduces to a Fourier transform in analogy to the full spectrum inversion method [6].

8 Conclusions

For a CT of type Φ_1 we have:
- the structure is symbolically $\Phi_1 \circ F \circ \Phi_{BP} u$
- both Φ_1 and F can be implemented by FFT-like algorithms
- the BP transformation reduces diffraction effects in vacuum

This should be compared with a CT of type Φ_2 which is characterized as follows:
- the structure is $\Phi_2 u$ (i.e., without back-propagation)
- simple structure, open whether Φ_2 has an FFT-like implementation
- diffraction effects in vacuum are not completely removed

Acknowledgements

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